PRICES OF BARRIER AND FIRST-TOUCH DIGITAL OPTIONS IN LÉVY-DRIVEN MODELS, NEAR BARRIER

MITYA BOYARCHENKO, MARCO DE INNOCENTIS AND SERGEI LEVENDORSKIĬ

Abstract. We calculate the leading term of asymptotics of the prices of barrier options and first touch digitals near the barrier for wide classes of Lévy processes with exponential jump densities, including Variance Gamma model, KoBoL (a.k.a. CGMY) model and Normal Inverse Gaussian processes. In the case of processes of infinite activity and finite variation, with the drift pointing from the barrier, we prove that the price is discontinuous at the boundary. In many cases, we calculate the second term of asymptotics as well.

Key words and phrases: Option pricing, barrier options, first-touch digitals, Lévy processes, Carr's randomization, KoBoL processes, CGMY model, Normal Inverse Gaussian processes, Variance Gamma processes, Wiener-Hopf factorization, asymptotics

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1. Introduction

1.1. The problem of pricing and hedging barrier options has attracted much attention in recent years, both from the theoretical finance side and from the practitioners' side. For instance, a rather comprehensive review of the 1965–1995 literature on the pricing of barrier options given in [15] lists about 30 articles, while hundreds of new works devoted to the same topic have appeared since 1995. In the framework of the Black-Scholes market model [5], an explicit formula for the price of a barrier call option was obtained by Merton [34]. Many subsequent works on barrier and first-touch digital options also remained in the Black-Scholes framework (the interested reader may wish to consult, for example, the bibliography lists in [15] and [14]).

However, it is a known fact that the Black-Scholes model yields rather inaccurate prices of barrier options near expiry, especially when the spot price S is close to the barrier. The reason is that the Black-Scholes value function $V_{BS} = V_{barr,BS}(T, S)$, where T is time to maturity, for (say) a down-and-out barrier put option is continuously differentiable with respect to S within the closed interval $[H, +\infty)$ whereas for many classes of Lévy processes, the delta is unbounded as the underlying tends to the barrier. The dashed line in Figure 1 is an example of such behavior if the underlying process is Brownian motion: we can clearly see how the delta, $\partial V_{BS}/\partial S$, of the option has a finite limit as S approaches H from the right. The line corresponding to the leading term of the asymptotics of the price as the price of underlying approaches the barrier is also shown; calculation of such asymptotics is the main object of the paper. Figure 2 is an enlarged version of Figure 1 near the barrier.

We see that the relative errors of prices of barrier options computed using the Black-Scholes model versus more realistic stock pricing models can sometimes reach several dozen percent near the barrier. This observation is not new – see, e.g., [24], also for NIG model; in this paper, we study possible shapes of the price curves near the boundary in a systematic manner.

Similar problems arise if the underlying pure jump Lévy process (or a Lévy process with insignificant diffusion component) is approximated by a jump-diffusion with a tangible diffusion component. The most convenient jump-diffusion models are the

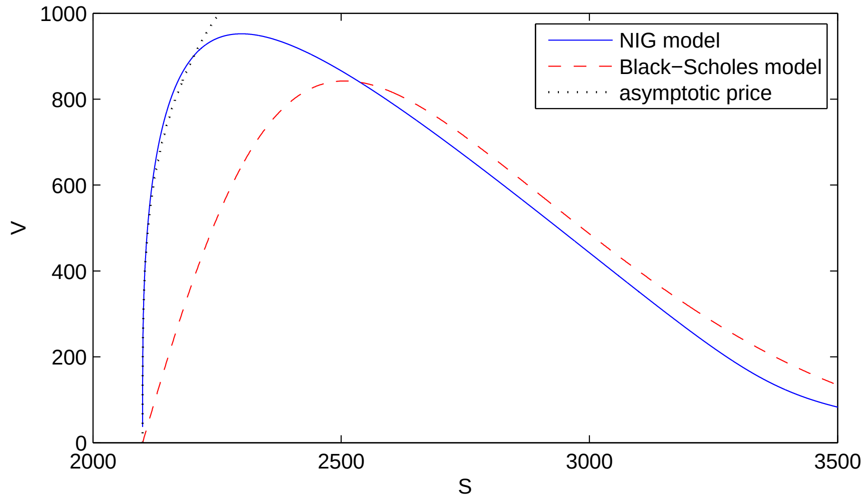

FIGURE 1. The value function of a down-and-out barrier put option in the NIG and Black-Scholes models. The strike price is $K = 3500$, the barrier is $H = 2100$, the time to maturity is $T = 0.25$ years, the riskless rate is 3%, and the underlying stock pays no dividends. (The example is taken from [23].)
Solid line: the graph of the value function calculated assuming that under a risk-neutral measure chosen by the market, the log-price process $\{X_t = \ln S_t\}$ of the underlying is a NIG process with parameters $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$.
Dashed line: same as above, except that X is assumed to be a Brownian motion with volatility $\sigma \approx 0.2136$, chosen so that the second (instantaneous) moment, σ^2, of $X = \{X_t\}$ is the same as the second (instantaneous) moment, $\delta\alpha^2(\alpha^2 - \beta^2)^{-3/2}$, of the NIG process in the first example.
Dotted line: the asymptotic price (see Appendix E).

double-exponential jump-diffusion model introduced by Kou [26] and its natural generalization constructed in [29] to price American options and labelled later *hyper-exponential jump-diffusion model* (HEJD) by Jeannin and Pistorius [23], who derived explicit formulas for the Laplace transforms with respect to the time variable of the value functions, deltas, gammas and thetas of the barrier options. They further showed that, approximating other Lévy-driven models (such as Variance Gamma [33, 32, 31] and NIG) by suitable hyper-exponential models, one can obtain accurate approximations to the prices and sensitivities of barrier and first-touch digital options in the regions not too close to the barrier. For a more recent and more detailed approach to an approximation technique of a similar nature, and for its application to the pricing of double barrier options, we refer the reader to [20] and [16], respectively.

Nevertheless, hyper-exponential models (with nonzero Gaussian component) also have the disadvantage that the value functions of barrier options in these models are continuously differentiable up to the barrier. In other words, qualitatively these functions exhibit behavior similar to that of the dashed line in Figure 1, whereas

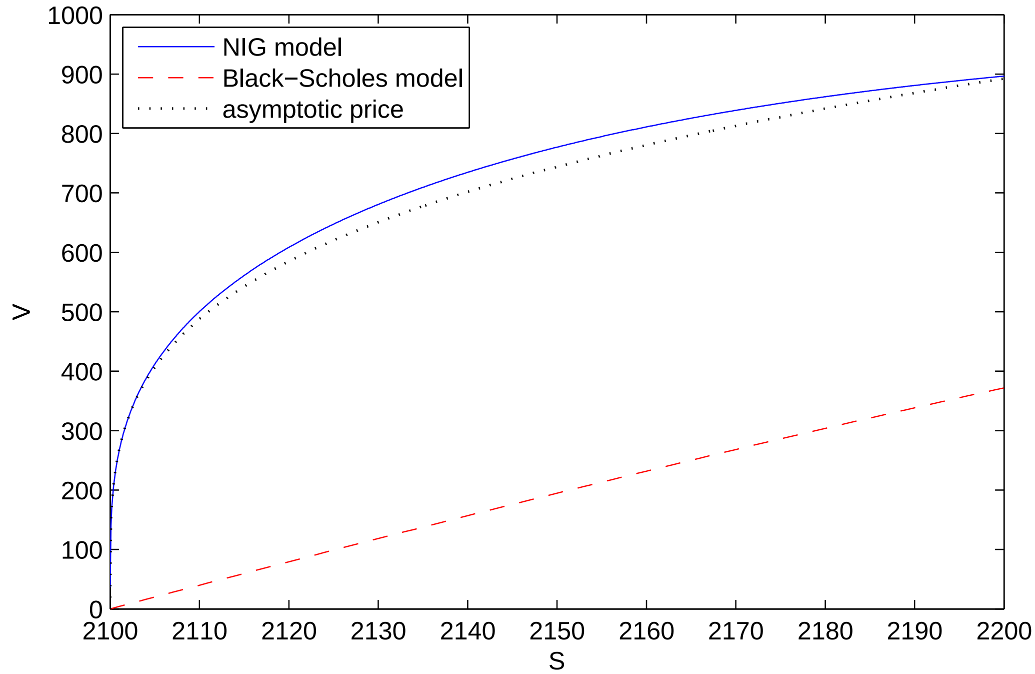

FIGURE 2. An enlarged version of Figure 1 near the barrier.

value functions obtained from more realistic models of stock prices, such as NIG and KoBoL, exhibit behavior similar to that of the solid line in Figure 1.

Other methods of pricing barrier options that use models with a tangible diffusion component to approximate models with zero diffusion component (such as the method of Cont and Voltchkova [19]) suffer from the same problem (see [30, 24] for an analysis of the errors that result from applying these methods). This issue is important because empirical studies of financial markets (see, e.g., [17]) show that, typically, the dynamics of a stock has zero diffusion component.

The discussion above is related to barrier options with continuous monitoring, and in the paper, we also consider options with continuous monitoring. For the review of results in the case of options with discrete monitoring, see, e.g., [22] and the bibliography therein.

The claims above about the non-standard behavior of prices of barrier options and first touch digitals near the boundary are based on the calculation (in [11, Ch. 7]) of the leading term of asymptotics of the value function of $V_{\mathrm{f.t.}}(T, S)$ of first touch digitals. In the infinite horizon case, the leading term of asymptotics for barrier options and first touch digitals was calculated in [10]. The claim about barrier options with finite time horizon was based on the observation (implied by the analytic intuition) that, qualitatively, the behavior of $V_{barr}(T,S)$ is similar to the behavior of $1-V_{\rm f.t.}(T,S)$; however, this fact was never proved to the best of our knowledge, and this remark does not give the coefficient in the asymptotic formulas. Furthermore, the asymptotics of the value function of the first touch digitals was calculated under a condition which excludes the Variance Gamma (VG) model introduced to finance by Madan with co-authors [33, 32, 31], and processes of finite variation, with non-zero drift. In particular, the processes of the extended Koponen's family[1] of order $\nu\in(0,1)$, with non-zero drift, were excluded.

1.2. **Main results.**

1.2.1. *Barrier options, leading term.* Let $X_t=\ln S_t$ be a Lévy process under an EMM chosen for pricing, let $G(X_T)$ be the payoff of the down-and-out barrier option at maturity, with barrier $H=1$, and let $V(T,x)$ be the price of the option. Let $\phi^\pm_q(\xi)$ be the Wiener-Hopf factors (WH-factors) in (C.5) and $\mathcal{E}^\pm_q$ be the EPV operators under the supremum and infimum processes (see (3.5)-(3.6)). Under weak regularity conditions on G and X, which are satisfied for puts, calls and digitals, and main classes of Lévy processes used in empirical studies of financial markets: KoBoL (a.k.a. CGMY model), NIG and its generalization, NTS model, and VG model, the exception being VG model with non-positive drift, we prove that there exists $\sigma>0$ such that

(a) function $\tilde G(q,0+)=(\mathcal{E}^+_qG)(0+)$ is well-defined and analytic in the half-plane $\mathrm{Re}\, q\ge\sigma$;

(b) for q in the half-plane $\mathrm{Re}\, q\ge\sigma$, the WH factors have the following asymptotics as $\xi\to\infty$ in a strip around the real axis:

$$\phi^\pm_q(\xi)=\phi^\pm_{q,\infty}(1\mp i\xi)^{-\nu_\pm}(1+O(|\xi|^{-\rho})), \tag{1.1}$$

where $\rho>0$, and functions $q\mapsto\phi^\pm_{q,\infty}$ are analytic in the half-plane $\mathrm{Re}\, q\ge\sigma$;

(c) there exist $C,s>0$ such that

$$|\partial_q(q^{-1}\phi^-_{q,\infty}\tilde G(q,0+))|\le C|q|^{-1-s},\quad \mathrm{Re}\, q\ge\sigma, \tag{1.2}$$

hence, the following integral absolutely converges:

$$\kappa(T)=\frac{e^{-rT}}{2\pi i\Gamma(1+\nu_-)(-T)}\int_{\mathrm{Re}\, q=\sigma}e^{qT}\partial_q(q^{-1}\phi^-_{q,\infty}\tilde G(q,0+))dq; \tag{1.3}$$

(d) there exists $s>0$ such that as $x\downarrow 0$,

$$V(T,x)=\kappa(T)x^{\nu_-}+O(x^{\nu_-+s}). \tag{1.4}$$

Explicit analytical formulas for $\phi^\pm_{q,\infty}$ and $\tilde G(q,0+)$ will be derived. The exponent $\nu_-\ge0$, which determines the rate of decay of the price near the barrier, is determined by the type of the process, and the coefficient $\kappa(T)>0$ depends on all parameters of the process and the type of the option; typically, $\nu_-\in(0,1)$, prominent exceptions

[1][9], a.k.a. name CGMY model [17] and KoBoL processes [11]; we adopt the latter terminology.

being processes with a Brownian motion component, and processes of finite variation with the drift pointing to the barrier, when $\nu_-=1$. For processes of finite variation with the drift pointing from the barrier, $\nu_-=0$, and, hence, the prices of the down-and-out option and first touch digital are discontinuous at the barrier.

1.2.2. *Two-term asymptotic formulas.* Under additional conditions on the parameters of the process, we derive two-term asymptotic formulas as well.

1.2.3. *First touch digital option.* Consider the option, which pays \$1 the first time the process X enters $(-\infty,0]$; if this does not happen until T, the option expires worthless. Similarly to (1.4), we prove that there exists $s>0$ such that as $x\downarrow 0$,

$$V(T,x)=1-\kappa_{\mathrm{f.t.}}(T)x^{\nu_-}+O(x^{\nu_-+s}). \tag{1.5}$$

where

$$\kappa_{\mathrm{f.t.}}(T)=\frac{e^{-rT}}{2\pi i\Gamma(1+\nu_-)(-T)}\int_{\mathrm{Re}\,q=\sigma}e^{qT}\partial_q((q-r)^{-1}\phi^-_{q,\infty})dq, \tag{1.6}$$

and $\sigma>r$ is arbitrary. Two-term asymptotic formulas can be derived for this case as well.

1.3. **Organization of the paper.** The rest of this text is organized as follows. In §2, we introduce the types of options and classes of processes considered in the paper. In §3, we present the well-known formulas that express the prices of barrier and first-touch digital options in probabilistic terms, calculate the Laplace transforms of these expressions w.r.t. time to maturity in terms of the expected presented value operators (EPV-operators) under supremum and infimum processes[2], write the formulas for the prices using the inverse Laplace transform, and explain the main idea for the calculation of the asymptotics. The asymptotic formulas are valid for the wide class of Lévy processes described in §2.3. The calculation is based on several basic facts of the theory of generalized functions collected in Appendix A, and on explicit formulas for the Wiener-Hopf factors, which we present in Appendix C. Details of calculation of the leading term, which needs case-by-case study, are presented, first, in §4 for processes of order $\nu\in(1,2)$, the main example being KoBoL processes (a.k.a. CGMY model) of infinite variation, and then in Appendices D and E for other cases, with explanations on why the variations are necessary. The two-term asymptotic formula is derived, first, in §5, for the case of processes of finite variations with the drift pointing from the boundary, next, in §6, for processes of infinite variation, and then, in §7, for processes of finite variation with the drift pointing toward the boundary. Numerical examples appear in §8; §9 concludes. Technical proofs are relegated to Appendix F.

[2]Here we use the operator form of the Wiener-Hopf factorization method developed for option pricing by S. Boyarchenko and Levendorskiĭ [10, 11, 12, 30, 13], with an important further development in [7]

2. Types of options and classes of Lévy processes

2.1. **Some types of options.** We begin by recalling the exercise rules for a few barrier and first-touch digital options that are commonly traded in financial markets.

A *down-and-out barrier call* (respectively, *put*) on a given asset is determined by three parameters: maturity date T, strike price K, and barrier H. The option expires worthless if, at any moment in time $t \le T$, the price, S_t, of the underlying reaches or falls below H. Otherwise, at time $t = T$, the owner of the option receives a payoff equal to $(S_T - K)_+ = \max\{S_T - K, 0\}$ (respectively, $(K - S_T)_+$).

An *up-and-out barrier call* (respectively, *put*) option is defined similarly, the only difference being that it expires worthless if S_t reaches or *exceeds* the barrier H for some $t \le T$. In principle, we do not have to assume a relationship between H and K, but we note that a down-and-out barrier put option (respectively, an up-and-out barrier call option) is worthless unless $H < K$ (respectively, $H > K$).

A *down-and-in* (respectively, *up-and-in*) *first-touch digital*[3] option is determined by its maturity date T and barrier H. The option pays its owner \$1 at the first moment in time $t \le T$ when the price, S_t, of the underlying reaches or falls below (respectively, reaches or exceeds) H. If no such event occurs, the option expires worthless at time T.

Remarks 2.1. (1) Down/up-and-out call/put barrier options are collectively referred to as *knock-out* barrier options. There are four other types of (single-)barrier options, namely, the *knock-in* ones, which become *activated* (as opposed to deactivated) when the price of the underlying crosses the barrier in the specified way. However, a package consisting of a knock-out barrier option and a knock-in barrier option with the same parameters clearly has value equal to that of a European option of the corresponding type. Thus, knowing the prices and sensitivities of knock-out barrier options and European options, one can calculate the prices and sensitivities of knock-in barrier options. Hence it is unnecessary to consider the latter type of options separately.

(2) It will be clear that our approach to calculating prices and sensitivities of down-and-out barrier put options (respectively, down-and-in digital options) can be easily modified to cover the other three types of knock-out barrier options (respectively, up-and-in digital options).

(3) A package consisting of a knock-out barrier option and a certain quantity of knock-in digital options with the same barrier is equivalent to a knock-out barrier option with constant rebate. Thus the latter options can also be easily studied using our techniques.

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[3] Sometimes (see, e.g., [23]) one refers to these options as *American down-and-in* (respectively, *up-and-in*) *digital* options. The reason is that an American-style option with a binary payoff function is always optimal to exercise at the first moment in time when the payoff becomes positive.

2.2. **Lévy-driven models.** For an exposition of the general theory of Lévy processes and their applications to pricing derivative securities, we refer the reader to [4, 36] and [11, 18, 37], respectively. We recall that every Lévy process $X = \{X_t\}_{t\geq 0}$ has the *characteristic exponent*, which is a continuous function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ satisfying $\psi(0) = 0$ and

$$\mathbb{E}\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)} \qquad \forall\, \xi \in \mathbb{R},\ t \geq 0;$$

and, conversely, the law of a Lévy process is uniquely determined by its characteristic exponent [36, Thm. 7.10]. Some examples of Lévy processes that are commonly used in empirical studies of financial markets are listed in §2.4 below.

We consider a model frictionless market consisting of a riskless bond and a stock, which is modelled as an exponential Lévy process $S_t = e^{X_t}$, under a chosen equivalent martingale measure (EMM) $\mathbb{Q}$. The riskless rate r is constant. We remark that, in general, $\mathbb{Q}$ is not unique. We assume that an EMM $\mathbb{Q}$ has been fixed once and for all, and all expectation operators appearing in this text will be with respect to this measure. The characteristic exponent ψ of X is also under this $\mathbb{Q}$.

If the stock does not pay dividends, then $e^{-rt}S_t$ must be a martingale under $\mathbb{Q}$. In terms of the characteristic exponent of the log-price process $\{X_t\}$, the EMM-condition can be written as follows: $r + \psi(-i) = 0$, where we are implicitly assuming that $\psi(\xi)$ admits the analytic continuation into the closed strip $-1 \leq \operatorname{Im}\xi \leq 0$ (if this is not the case, then $\mathbb{E}[S_t] = \infty$ for all $t > 0$, i.e., the process $\{S_t\}$ cannot be priced; we exclude this situation from our consideration). If the stock pays dividends at constant rate δ, then the EMM condition becomes $r + \psi(-i) = \delta$.

2.3. **Strongly regular Lévy processes of exponential type.** The main classes of processes used in the study of financial markets (see the examples in the next subsection) enjoy the following property: there exist $\lambda_- < -1 < 0 < \lambda_+$ such that the underlying Lévy process X is of exponential type (λ_-, λ_+). This means that the characteristic exponent, $\psi(\xi)$, of X, admits the analytic continuation into the open strip $\operatorname{Im}\xi \in (\lambda_-, \lambda_+)$. Moreover, $\psi(\xi)$ grows at most polynomially as $\operatorname{Re}\xi \to \pm\infty$ within every closed strip $\operatorname{Im}\xi \in [\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$. For the behavior of the value function (especially near the barrier) to be sufficiently regular, and for the leading term of the asymptotics of a simple form to exist, we need additional regularity conditions. In [11, Definition 3.2], such conditions were formulated, and the resulting class of processes was called Regular Lévy Processes of Exponential type (RLPE) in 1D. Two definitions were given: in terms of the properties of the Lévy measure and in terms of the Lévy exponent. Loosely speaking, the first definition postulates that the Lévy measure behaves, asymptotically, as $c_\pm|x|^{-\nu-1}$ as $x \to \pm 0$, and decays at least as fast as $e^{-\lambda^\pm x}$ as $x \to \mp\infty$. An almost equivalent definition in terms of the characteristic exponent is as follows

Definition 2.2. Let $\lambda_- < 0 < \lambda_+$ and $\nu \in (0, 2]$. We call X a *regular Lévy process of exponential type* (λ_-, λ_+) and *order* ν if its characteristic exponent admits the analytic continuation in the strip $\operatorname{Im}\xi \in (\lambda_-, \lambda_+)$, and, for some $\mu \in \mathbb{R}$, function $\psi(\xi) + i\mu\xi$ is asymptotically positively homogeneous of order ν as $\xi \to \pm\infty$ in any strip $\operatorname{Im}\xi \in [\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$.

In this paper, we will use somewhat stronger conditions, which simplify the proofs.

Definition 2.3. Let $\lambda_- < 0 < \lambda_+$ and $\nu \in (0, 2]$. We call X a *strongly regular Lévy process of exponential type* (λ_-, λ_+) and *order* ν (sRLPE of order ν) if the following conditions hold:

(i) the characteristic exponent ψ admits the analytic continuation into the complex plane with two cuts $i(-\infty, \lambda_-]$ and $i[\lambda_+, +\infty)$;
(ii) for $z < \lambda_-$ and $z > \lambda_+$, the limits $\psi(iz \pm 0)$ exist;
(iii) there exists $\mu \in \mathbb{R}$ such that the function $\psi^0(\xi) := \psi(\xi) + i\mu\xi$ is asymptotically positively homogeneous of order ν as $\xi \to \infty$ in the complex plane with these cuts;
(iv) to be more specific, there exist $\nu_1 < \nu$ and $d^0_\pm$, $\operatorname{Re} d^0_\pm > 0$, such that as $\rho \to +\infty$,

$$\psi^0(\rho e^{i\varphi}) = d^0_+ e^{i\varphi\nu}\rho^\nu + O(\rho^{\nu_1}), \quad \varphi \in [0, \pi/2 - 0]; \tag{2.1}$$

$$\psi^0(\rho e^{i\varphi}) = d^0_- e^{i(-\pi+\varphi)\nu}\rho^\nu + O(\rho^{\nu_1}), \quad \varphi \in [\pi/2 + 0, \pi]; \tag{2.2}$$

$$\psi^0(\rho e^{i\varphi}) = d^0_- e^{i(\pi+\varphi)\nu}\rho^\nu + O(\rho^{\nu_1}), \quad \varphi \in [-\pi, -\pi/2 - 0]; \tag{2.3}$$

$$\psi^0(\rho e^{i\varphi}) = d^0_+ e^{i\varphi\nu}\rho^\nu + O(\rho^{\nu_1}), \quad \varphi \in [-\pi/2 + 0, 0]. \tag{2.4}$$

The notation $\varphi = \pi/2 \pm 0$ means that $\eta = \rho e^{i\varphi}$ is of the form $\eta = iz \mp 0$, where $z > 0$, and $\varphi = -\pi/2 \pm 0$ means that $\eta = \rho e^{i\varphi}$ is of the form $\eta = iz \pm 0$, where $z < 0$.

2.4. **Model classes.**

(1) A Brownian motion (used in the classical Black-Scholes model [5]) is an sRLPE of order 2 and exponential type $(-\infty, \infty)$. Its characteristic exponent is given by $\frac{\sigma^2}{2}\xi^2 - i\mu\xi$, where $\sigma > 0$ is the volatility and $\mu \in \mathbb{R}$ is the drift of the process. We have $d^0_\pm = \sigma^2/2$.

(2) Kou's model [26], its generalization constructed in [29] and labelled later a hyper-exponential jump-diffusion (HEJD) [23] are sRLPEs of order 2, and $d^0_\pm = \sigma^2/2$ is the same as in the BM case.

(3) Lévy processes of the *extended Koponen family* (generalizing the class of processes introduced by Koponen [25]) were defined by S. Boyarchenko and Levendorskiĭ in [9]. Later the same family of Lévy processes was used in [17] under the name "CGMY-model," and in [11] under the name "KoBoL processes." We adopt the latter terminology. The characteristic exponent of a KoBoL process of order $\nu \in (0, 2)$, $\nu \neq 1$, is of the form

$$\psi(\xi) = -i\mu\xi + \Gamma(-\nu) \cdot \left[c_+((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu) + c_-(\lambda_+^\nu - (\lambda_+ + i\xi)^\nu)\right], \tag{2.5}$$

where $\lambda_- < 0 < \lambda_+$ are called the *steepness parameters* of the process, $c_\pm > 0$ characterize the *intensity* of positive and negative jumps, and $\mu \in \mathbb{R}$. (If $\nu < 1$, then μ is the drift). In (2.5) and elsewhere in the text, we use the standard convention that $z^\nu = e^{\nu \cdot \ln z}$ for any $\nu \in \mathbb{C}$ and any $z \in \mathbb{C}$ such that $z \notin (-\infty, 0]$. In turn, $\ln z$ denotes the unique branch of the natural logarithm function defined on the complex plane with the negative real axis $(-\infty, 0]$ removed, determined by the requirement that $\ln(1) = 0$.

A KoBoL process with parameters as above has exponential type (λ_-, λ_+), so there is no conflict of notation.[4] We have

$$d^0_\pm = -\Gamma(-\nu)[c_+(\mp i)^\nu + c_-(\pm i)^\nu]. \tag{2.6}$$

(4) Variance Gamma (VG) processes were first used in empirical studies of financial markets by Dilip Madan and collaborators [33, 32, 31]. The characteristic exponent of a VG process is of the form[5]:

$$\psi(\xi) = -i\mu\xi + c_+[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)] + c_-[\ln(\lambda_+ + i\xi) - \ln(\lambda_+)], \tag{2.7}$$

where $\lambda_- < 0 < \lambda_+$, $c > 0$ and $\mu \in \mathbb{R}$. A VG process with these parameters is also a Lévy process of exponential type (λ_-, λ_+); but it is not an RLPE because the jump term in (2.7) increases at infinity as a logarithm.

(5) Normal Inverse Gaussian (NIG) processes were introduced by Barndorff-Nielsen [2]. A natural generalization of NIG was constructed in [3] and called *Normal Tempered Stable Lévy processes* (NTS). The characteristic exponent of an NTS process is of the form

$$\psi(\xi) = -i\mu\xi + \delta \cdot \left[\left(\alpha^2 - (\beta + i\xi)^2\right)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}\right], \tag{2.8}$$

where $\nu \in (0, 2)$, $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. An NTS process with these parameters has exponential type $(\beta - \alpha, \beta + \alpha)$, order ν and $d^0_\pm = \delta$. NIG is an NTS of order $\nu = 1$.

The verification in the case of BM, Kou's model and HEJD model is straightforward. In the case of KoBoL (a.k.a. CGMY) model, NTS and VG model, the verification of (i)–(ii) is straightforward as well. The verification of (iii)-(iv) for KoBoL and NTS is also straightforward but requires case-by-case study for $\varphi \in [\pi/2 - 0, 0]$, $\varphi \in [\pi, \pi/2 + 0]$, $\varphi \in [-\pi, -\pi/2 - 0]$ and $\varphi \in [-\pi/2 + 0, 0]$. All 4 cases being similar, we consider the third one. Let $a > 0$, $\nu \in (0, 2)$ be fixed, and $\xi = \rho e^{i\varphi}$, where $\varphi \in [-\pi, -\pi/2 - 0]$. Then, as $\rho \to +\infty$,

$$(a - i\xi)^\nu = (a - i\rho e^{i\varphi})^\nu = \left(a + \rho e^{i\left(\frac{3\pi}{2} + \varphi\right)}\right)^\nu \sim \rho^\nu e^{i\left(\frac{3\pi}{2} + \varphi\right)\nu}. \tag{2.9}$$

[4] In [9, 11], the reader can find a more general version of KoBoL, where the parts of the characteristic exponent responsible for positive and negative jumps have different orders; the case $\nu = 1$ is also considered, with the characteristic exponent given by a formula different from (2.5).

[5] What we present is not the most common way of writing the formula. Rather, we chose an expression that is equivalent to the standard one and makes the analogy with (2.5) transparent.

$$(a+i\xi)^\nu = (a+i\rho e^{i\varphi})^\nu = \left(a+\rho e^{i\left(\frac{\pi}{2}+\varphi\right)}\right)^\nu \sim \rho^\nu e^{i\left(\frac{\pi}{2}+\varphi\right)\nu}, \tag{2.10}$$

and (2.3) for KoBoL and NTS is immediate. For the VG model, we have

$$\ln(a-i\xi) = \ln\left(a+\rho e^{i\left(\frac{3\pi}{2}+\varphi\right)}\right) \sim \ln\rho + i\left(\frac{3\pi}{2}+\varphi\right). \tag{2.11}$$

$$\ln(a+i\xi) = \ln\left(a+\rho e^{i\left(\frac{\pi}{2}+\varphi\right)}\right) \sim \ln\rho + i\left(\frac{\pi}{2}+\varphi\right). \tag{2.12}$$

Non-power asymptotic behavior in (2.11)–(2.12) leads to certain complications, when we calculate the asymptotics of the price as $x = \ln S \to h = \ln H$. Nevertheless, some of the results, which we will derive for processes of order $\nu \in (0,1)$, will hold for the VG model as well. We label the VG model *an sRLPE of order* $0+$.

Hyperbolic processes are sRLPEs of order 1 but the verification is not so easy. See [10], where the asymptotic properties of the characteristic exponents of model classes where studied and used for different purposes.

2.5. **Important constants characterizing an sRLPE.** For real ξ, $\overline{\psi^0(\xi)} = \psi^0(-\xi)$, hence, $\overline{d^0_-} = d^0_+$. We consider the following cases:

(i) if $\nu \in (1,2]$ or $\nu \in (0,1)$ and $\mu = 0$, set $d_\pm = d^0_\pm$, $d = |d_\pm|$, $\gamma_\pm = \arg d_\pm$, $\bar\nu = \nu$,

$$\nu_\pm = \nu/2 - \gamma_\pm/\pi; \tag{2.13}$$

(ii) if $\nu = 1$, set $d_\pm = \mp i\mu + d^0_\pm$, $d = |d_\pm|$, $\gamma_\pm = \arg d_\pm$, $\nu_\pm = 1/2 - \gamma_\pm/\pi$, $\bar\nu = 1$;
(iii) if $\nu \in [0+,1)$ and $\mu > 0$, set $d_\pm = \mp i\mu$, $\nu_+ = 1$, $\nu_- = 0$, $\bar\nu = 1$;
(iv) if $\nu \in [0+,1)$ and $\mu < 0$, set $d_\pm = \mp i\mu$, $\nu_+ = 0$, $\nu_- = 1$, $\bar\nu = 1$.

Notice that in all cases, $\nu_+ + \nu_- = \bar\nu$, and

$$\nu_\pm = \bar\nu/2 - \gamma_\pm/\pi. \tag{2.14}$$

Furthermore, $\nu_\pm \in (0,\nu)$ if $\nu \ge 1$; if $\nu \in (0,1)$ and $\mu = 0$, then we require that $\gamma := |\gamma_\pm|$ be in $[0, \pi\nu/2)$; then $\nu_\pm \in (0,\nu)$ as well.

2.6. **Asymptotics of $\psi(\xi)$.** It follows from (i)–(iv) and (2.1)–(2.4), that, if $\nu \in (0,2]$, or $\nu = 0+$ and $\mu \neq 0$, then there exists $\nu_1 < \bar\nu$ such that as $\rho \to +\infty$,

$$\psi(\rho e^{i\varphi}) = d_+ e^{i\varphi\bar\nu}\rho^{\bar\nu} + O(\rho^{\nu_1}), \quad \varphi \in [0, \pi/2 - 0]; \tag{2.15}$$

$$\psi(\rho e^{i\varphi}) = d_- e^{i(-\pi+\varphi)\bar\nu}\rho^{\bar\nu} + O(\rho^{\nu_1}), \quad \varphi \in [\pi/2 + 0, \pi]; \tag{2.16}$$

$$\psi(\rho e^{i\varphi}) = d_- e^{i(\pi+\varphi)\bar\nu}\rho^{\bar\nu} + O(\rho^{\nu_1}), \quad \varphi \in [-\pi, -\pi/2 - 0]; \tag{2.17}$$

$$\psi(\rho e^{i\varphi}) = d_+ e^{i\varphi\bar\nu}\rho^{\bar\nu} + O(\rho^{\nu_1}), \quad \varphi \in [-\pi/2 + 0, 0]. \tag{2.18}$$

Lemma 2.4. *In each of Cases (i)–(iv), there exists $\beta_0 > 0$ such that for any $a > 0$,*

$$\frac{\psi(\xi)}{d(a-i\xi)^{\nu_+}(a+i\xi)^{\nu_-}} = 1 + O(|\xi|^{-\beta_0}) \tag{2.19}$$

as $\xi \to \infty$ in the complex plane with the cuts $i(-\infty, \lambda_-]$, $i[\lambda_+, +\infty)$.

Remark 2.5. For model classes, we have: if $\nu > 1$ and $\mu \neq 0$, then $\beta_0 = \nu - 1$; if $\nu \in [0+, 1)$ and $\mu \neq 0$, then $\beta_0 = 1 - \nu$; if $\nu = 1$, then $\beta_0 = 1$; if $\nu > 1$ and $\mu = 0$, then $\beta_0 = 1$; if $\nu \in (0, 1)$ and $\mu = 0$, then $\beta_0 = -\nu$.

Proof. The verification is straightforward but requires case-by-case study for $\varphi \in [\pi/2 - 0, 0]$, $\varphi \in [\pi, \pi/2 + 0]$, $\varphi \in [-\pi, -\pi/2 - 0]$ and $\varphi \in [-\pi/2 + 0, 0]$. All 4 cases being similar, we consider the third one. It follows from (2.9)–(2.10) that

$$d(a - i\xi)^{\nu_+}(a + i\xi)^{\nu_-} = d\rho^{\nu_- + \nu_+} e^{i\left[\left(\frac{3\pi}{2} + \varphi\right)\nu_+ + \left(\frac{\pi}{2} + \varphi\right)\nu_-\right]} + O(\rho^{\bar{\nu} - 1}).$$

Since

$$\left(\frac{3\pi}{2} + \varphi\right)\nu_+ + \left(\frac{\pi}{2} + \varphi\right)\nu_- = (\pi + \varphi)\bar{\nu} - \frac{\pi}{2}(\nu_- - \nu_+) = (\pi + \varphi)\bar{\nu} + \gamma_-,$$

we have

$$d(a - i\xi)^{\nu_+}(a + i\xi)^{\nu_-} = d_- e^{i(\pi + \varphi)\bar{\nu}}\rho^{\bar{\nu}} + O(\rho^{\bar{\nu} - 1}).$$

Comparing with (2.17), we conclude that (2.19) holds if $\nu > 0$. If $\nu = 0+$ and $\mu \neq 0$, then we take (2.11)–(2.12) into account, and obtain (2.19) for any $\beta_0 < 1$. □

3. General formulas for barrier options and first touch digitals

3.1. **Down-and-out options.** In this subsection, we consider a down-and-out barrier contingent claim with barrier H, expiry date T, and terminal payoff function $G(x)$, which is a nonnegative measurable function on $\mathbb{R}$. If, at any time $t \le T$ prior to expiry, $S_t = e^{X_t}$, the price of the underlying, reaches or falls below $H = e^h$, the claim expires worthless. Otherwise, at expiry, the claim yields a payoff equal to $G(X_T)$. The payoff function is non-negative and measurable. Contingent claims of this type provide a common generalization of down-and-out barrier call and put options. In particular, $G(x) = (e^x - K)_+$ (respectively, $G(x) = (K - e^x)_+$) for a down-and-out barrier call (respectively, put) option, and $G(x) = \mathbb{1}_{[\ln K, +\infty)}(x)$ (respectively, $G(x) = \mathbb{1}_{(-\infty, \ln K]}(x)$) for the digital call (respectively, put), with strike price $K > H$. For simplicity, we impose an additional regularity condition on the Fourier transform of the payoff function, which is satisfied for down-and-out barrier (European) call and put options:

$$|\hat{G}(\xi)| \le C(1 + |\xi|)^{-2} \tag{3.1}$$

along any line $\operatorname{Im}\xi = \omega$, with the exception of a finite number of omegas (indeed, for puts and calls, $\hat{G}(\xi)$ has two poles at $\xi = 0, -i$). Condition (3.1) excludes digital down-and-out options with strike $K > H$ and digital no-touch option with $G = \mathbb{1}_{\mathbb{R}}$. The latter case is, in fact, simper than the cases of the put and call options, as it will be seen from the proof. In Appendix B, it will be explained how the proofs should be modified to incorporate digital puts and call.

Finally, pricing the call option with $K < H$ reduces pricing of the digital no-touch option by an appropriate change of measure.

The time-0 value of the barrier option with payoff $G(X_T)$ is given by

$$V(T,x) = \mathbb{E}^x\left[e^{-rT}G(X_T)\mathbb{1}_{\{\tau_h^- > T\}}\right], \tag{3.2}$$

where x is the current log-spot price of the underlying, $h = \ln H$, and τ_h^- is the first entrance time of the process $\{X_t\}_{t\ge 0}$ into the interval $(-\infty, h]$.

As it is well-known, $\hat{V}(q,x)$, the Laplace transform of $V(T,x)$, can be calculated more explicitly. Applying Fubini's theorem, we obtain that $\hat{V}(q,x)$ is the value function of the perpetual stream $G(X_t)$, which is terminated the first moment X_t reaches the barrier or falls below it, the discounting factor being $q+r$:

$$\hat{V}(q,x) \;=\; \int_0^{+\infty} e^{-qt}\mathbb{E}^x\left[e^{-rt}G(X_t)\mathbb{1}_{\{\tau_h^- > t\}}\right]dt = \mathbb{E}^x\left[\int_0^{\tau_h^-} e^{-(q+r)t}G(X_t)dt\right].$$

Before giving a formula for the last expectation, let us introduce some notation. First, we define the *supremum process* $\overline{X}$ and the *infimum process* $\underline{X}$ of X by

$$\overline{X}_t = \sup_{0\le s\le t} X_s, \qquad \underline{X}_t = \inf_{0\le s\le t} X_s. \tag{3.3}$$

Given any $q > 0$, we let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable with mean q^{-1}, and we define operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ acting on a nonnegative measurable (or an arbitrary bounded measurable) function f on $\mathbb{R}$ as follows:

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}^x\left[f(\overline{X}_{T_q})\right], \qquad (\mathcal{E}_q^- f)(x) = \mathbb{E}^x\left[f(\underline{X}_{T_q})\right]. \tag{3.4}$$

Writing explicitly the RHSs in (3.4) as expectations of integrals

$$(\mathcal{E}_q^+ f)(x) \;=\; \mathbb{E}^x\left[\int_0^{+\infty} qe^{-qt}f(\overline{X}_t)dt\right], \tag{3.5}$$

$$(\mathcal{E}_q^- f)(x) \;=\; \mathbb{E}^x\left[\int_0^{+\infty} qe^{-qt}f(\underline{X}_t)dt\right], \tag{3.6}$$

we obtain the interpretation of $\mathcal{E}_q^\pm$ as the expected present value operators (EPV-operators), which calculate the expected present value of the stream, under the supremum and infimum processes. Note that the EPV-operator $\mathcal{E}_q f(x) = \mathbb{E}^x\left[f(X_{T_q})\right]$ under the initial process is the normalized resolvent.

Lemma 3.1. *We have*

$$\hat{V}(q,x) = (q+r)^{-1}\left(\mathcal{E}_{q+r}^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_{q+r}^+ G\right)(x). \tag{3.7}$$

Equation (3.7) was derived in [11] for regular Lévy processes of exponential type, and in [7] for any Lévy process.

It is evident from (3.5) and (3.6) that the EPV-operators $\mathcal{E}^\pm_{q+r}$ admit the analytic continuation into the open half-plane $\mathrm{Re}\, q > -r$, which is uniformly bounded (as a family of operators in $L^\infty(\mathbb{R})$) in each closed half-plane $\mathrm{Re}\, q + r \ge \sigma > 0$. Hence, $\hat{V}(q, x)$ admits the bound

$$|\hat{V}(q,x)| \le C_\sigma |q+r|^{-1}, \quad \mathrm{Re}\, q + r \ge \sigma, \tag{3.8}$$

where C_σ depends on $\sigma > 0$ but not on q. Now, one can use the inverse Laplace transform and calculate the value function of the barrier option. To simplify the result, we make the change of variable $q + r \mapsto q$:

$$V(T,x) = \frac{e^{-rT}}{2\pi i} \int_{\mathrm{Re}\, q=\sigma} e^{qT} q^{-1} \left(\mathcal{E}^-_q \mathbb{1}_{(h,+\infty)} \mathcal{E}^+_q G\right)(x) dq. \tag{3.9}$$

To prove that the RHS in (3.9) is continuous as a function of $T > 0$, and derive the leading term of asymptotics, we integrate by parts k times using $e^{qT}dq = T^{-1}de^{qT}$:

$$V(T,x) \;=\; \frac{e^{-rT}}{2\pi i (-T)^k} \int_{\mathrm{Re}\, q=\sigma} e^{qT} \partial^k_q \left(q^{-1}\mathcal{E}^-_q \mathbb{1}_{(h,+\infty)} \mathcal{E}^+_q G\right)(x) dq. \tag{3.10}$$

(In analysis, one says that the integral in (3.9) is understood as an oscillatory integral). For the wide class of Lévy processes described in §2.3, we prove that the integrand on the RHS of (3.10) admits a bound via $C|q|^{-1-a}$, for some $a > 0$. For positive q, this estimate with $a = 1$ is immediate from (3.5)-(3.6) (differentiate under the integral sign and change the variable $t' = qt$), but for complex q, the proof is based on explicit formulas for the Wiener-Hopf factors $\phi^\pm_q(\xi)$, which we present in Appendix C. We conclude that the integral in (3.10) converges absolutely and uniformly in $T \ge 0$, and, therefore, $V(T, x)$ is a continuous function of $T > 0$.

3.2. **Idea of calculation of the asymptotics of the price as x approaches the boundary $h = 0$.** We systematically use elements of the theory of generalized functions collected for the reader's convenience in Appendix A and representation $\mathcal{E}^\pm_q = \phi^\pm_q(D)$ of the EPV operators as pseudo-differential operators with the symbols $\phi^\pm_q(\xi)$ (see Appendix C). The starting points is (A.2) with $u = \tilde{G} := \mathcal{E}^+_q G = \phi^+_q(D)G$ and $m = 1$:

$$\begin{aligned} \mathbb{1}_{(0,+\infty)}\tilde{G}(q,\cdot) \;&=\; \tilde{G}(q,0+)(1+iD)^{-1}\delta \\ &\quad +(1+iD)^{-1}\mathbb{1}_{(0,+\infty)}(1+iD)\tilde{G}(q,\cdot). \end{aligned} \tag{3.11}$$

Equation (3.11) is justified if, for each q of interest, $(1+iD)\tilde{G}(q,x) \in H^s(\mathbb{R})$ for some $s > -1/2$. It will follow from (3.1) and estimates for $\phi^+_q(\xi)$, which we will derive, that one can take any $s < 1/2$. Then, for any $\epsilon > 0$, the first and second term in (3.11) belong to $\mathring{H}^{1/2-\epsilon}(\mathbb{R}_+)$ and $\mathring{H}^{3/2-\epsilon}(\mathbb{R}_+)$, respectively. We substitute (3.11) into

(3.10) and consider the expression under the integral sign, before ∂_q^k is applied:

$$\begin{aligned} f(q,\cdot) &:= q^{-1}\phi_q^-(D)\mathbb{1}_{(0,+\infty)}\tilde{G}(q,\cdot) \\ &= q^{-1}\tilde{G}(q,0+)\phi_q^-(D)(1+iD)^{-1}\delta \\ &\quad +q^{-1}\phi_q^-(D)(1+iD)^{-1}\mathbb{1}_{(0,+\infty)}(1+iD)\tilde{G}(q,\cdot). \end{aligned} \tag{3.12}$$

On the strength of (1.1), the symbol $\phi_q^-(\xi)$ of a PDO $\phi_q^-(D) = \mathcal{E}_q^-$ decreases as $|\xi|^{-\nu_-}$ as $\xi \to \infty$ in the lower half-plane. This implies that, for any s, $\phi_q^-(D) : \mathring{H}^s(\mathbb{R}_+) \to \mathring{H}^{s+\nu_-}(\mathbb{R}_+)$ is bounded, therefore, $\phi_q^-(D)(1+iD)^{-1}\delta \in \mathring{H}^{\nu_-+1/2-\epsilon}(\mathbb{R}_+)$, and

$$\mathcal{E}_q^-(1+iD)^{-1}\mathbb{1}_{(0,+\infty)}(1+iD)\tilde{G}(q,\cdot) \in \mathring{H}^{\nu_-+3/2-\epsilon}(\mathbb{R}_+). \tag{3.13}$$

It follows from the Sobolev embedding theorem that, for any $\epsilon > 0$, the function in (3.13) is $O(x^{\nu_-+1-\epsilon})$, and $(\phi_q^-(D)(1+iD)^{-1})\delta(x) = O(x^{\nu_- -\epsilon})$ as $x \downarrow 0$. Furthermore, a more accurate study allows us to prove that

$$(\mathcal{E}_q^-(1+iD)^{-1}\delta)(x) \sim \Gamma(1+\nu_-)^{-1}\phi_{q,\infty}^- x^{\nu_-}, \tag{3.14}$$

where $\phi_{q,\infty}^- > 0$ is the asymptotic coefficient in (1.1), therefore, it is natural to expect that the second term in (3.12) can be included in the O-term of the asymptotic formula. To make this heuristic argument precise, we need to derive appropriate estimates, which are uniform in q in the half-plane $\operatorname{Re} q \ge \sigma$, and, moreover, admit integration.

We will prove that this idea works after integration by parts, which helps because with each differentiation w.r.t. q, the functions involved decay faster (as $q \to \infty$) than before differentiation; the rate of decay w.r.t. ξ either remains the same or increases. In the result, we are able to prove that, if k is large enough, then there exist $C, \sigma, \rho, s > 0$ such that, as $x \downarrow 0$, uniformly w.r.t. q in the half-plane $\operatorname{Re} q \ge \sigma$,

$$\partial_q^k \left(q^{-1}\mathcal{E}_q^-\mathbb{1}_{(0,+\infty)}\mathcal{E}_q^+\right) G(x) = \frac{\partial_q^k \left(q^{-1}\phi_{q,\infty}^-\tilde{G}(q,0+)\right)}{\Gamma(1+\nu_-)} x^{\nu_-} + R(k,q,x), \tag{3.15}$$

where

$$|R(k,q,x)| \le C|q|^{-1-\rho}x^{\nu_-+s}. \tag{3.16}$$

It follows that

$$V(T,x) = \kappa(T)x^{\nu_-} + O(x^{\nu_-+s}), \tag{3.17}$$

where

$$\kappa(T) = \frac{e^{-rT}}{2\pi i\Gamma(1+\nu_-)(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{qT}\partial_q^k \left(q^{-1}\phi_{q,\infty}^-\tilde{G}(q,0+)\right) dq. \tag{3.18}$$

We will also prove that for any $k \ge 1$, there exist $C, s > 0$ such that

$$|\partial_q^k(q^{-1}\phi_{q,\infty}^-\tilde{G}(q,0+))| \le C|q|^{-1-s}, \quad \operatorname{Re} q \ge \sigma, \tag{3.19}$$

hence, we can integrate by parts in (3.18) back and obtain (3.17)–(3.18) with $k = 1$.

In some cases, we will show that the EPV-operators $\mathcal{E}^{\pm}_q$ admit the analytic continuation into an obtuse sector of the form $\Sigma_{\sigma,\theta} = \{q = \sigma + \rho e^{i\varphi} \mid \varphi \in [-\theta, \theta]\}$, for some $\theta \in (\pi/2, \pi)$, and the bound (1.2) holds for $q \in \Sigma_{\sigma,\theta}$. Hence, we can deform the line of integration in (3.18) and obtain

$$\kappa(T) = \frac{e^{-rT}}{2\pi i \Gamma(1+\nu_-)(-T)^k} \int_{\partial\Sigma_{\sigma,\theta}} e^{qT} \partial_q^k \left(q^{-1} \phi^-_{q,\infty} \tilde{G}(q, 0+) \right) dq. \tag{3.20}$$

Using (1.2) and the fact that e^{qT} decreases exponentially as $q \to \infty$ along $\Sigma_{\sigma,\theta}$, we can integrate by parts back and get (3.20) with $k = 0$.

To calculate the second term of asymptotics, we start with (A.2) with $m = 2$, and prove that all terms but the first one give contribution of order $x^{\nu_- + 1 - \epsilon}$, for any $\epsilon > 0$. If $\nu \in (0, 2), \nu \neq 1$, and $\mu \neq 0$, we are able to prove that

$$(\mathcal{E}^-_q (1 + iD)^{-1} \delta)(x) \sim \Gamma(1 + \nu_-)^{-1} \phi^-_{q,\infty} x^{\nu_-} + c(q) x^{\nu_- + |\nu - 1|}, \tag{3.21}$$

where function $c(q)$ depends on ν and μ. Since $0 < |\nu - 1| < 1$, the second term in (3.21) gives rise to the second term of asymptotics of the price.

3.3. **First-touch digitals.** In the same situation as before, let us now consider a down-and-in first-touch digital option with maturity date T and barrier H. Let $V(T, x)$ denote the value of this option at time $t = 0$, where x is the current log-spot price of the underlying. Then

$$V(T, x) = \mathbb{E}^x \left[e^{-r\tau_h} \mathbb{1}_{\{\tau_h \le T\}} \right]. \tag{3.22}$$

As in the case of the down-and-out option, we calculate first $\hat{V}(q, x)$, the Laplace transform of $V(T, x)$.

Lemma 3.2. *We have* $\hat{V}(q, x) = q^{-1} (\mathcal{E}^-_{q+r} \mathbb{1}_{(-\infty, h]})(x)$.

Proof. In view of (3.22), we have

$$\hat{V}(q, x) := \mathbb{E}^x \left[\int_0^\infty e^{-r\tau_h} e^{-qt} \mathbb{1}_{\{\tau_h \le t\}} \, dt \right] = \mathbb{E}^x \left[e^{-r\tau_h} \int_{\tau_h}^\infty e^{-qt} \, dt \right] = q^{-1} \mathbb{E}^x \left[e^{-(q+r)\tau_h} \right].$$

Now $e^{-(q+r)\tau_h}$ is a random variable with values in $[0, 1]$. Introducing an auxiliary variable $\lambda \in [0, 1]$, we can write $\mathbb{E}^x \left[e^{-(q+r)\tau_h} \right] = \int_0^1 \mathbb{E}^x \left[\mathbb{1}_{\{e^{-(q+r)\tau_h} \ge \lambda\}} \right] d\lambda$. Next, we make the change of variables $t = -(\ln \lambda)/(q + r)$ in the last integral (so that $\lambda = e^{-(q+r)t}$ and $d\lambda = -(q + r) e^{-(q+r)t} dt$), which transforms it into

$$\int_0^\infty (q + r) e^{-(q+r)t} \mathbb{E}^x \left[\mathbb{1}_{\{\tau_h \le t\}} \right] dt.$$

Using Fubini's theorem and the fact that $\tau_h \le t$ if and only if $\underline{X}_t \le h$, we can rewrite the last integral as

$$\mathbb{E}^x\left[\int_0^\infty (q+r)e^{-(q+r)t}\mathbb{1}_{\{\underline{X}_t\le h\}}\,dt\right] = \mathbb{E}^x\left[\mathbb{1}_{\{\underline{X}_{T_{q+r}}\le h\}}\right] \stackrel{\text{def}}{=} \mathcal{E}^-_{q+r}\big(\mathbb{1}_{(-\infty,h]}(x)\big),$$

which proves Lemma 3.2. □

Taking the inverse Laplace transform, and then using $\mathbb{1}_{(-\infty,h]} = \mathbb{1} - \mathbb{1}_{(h,+\infty)}$ and the residue theorem, we obtain, for $\sigma > r$,

$$\begin{aligned} V(T,x) &= \frac{e^{-rT}}{2\pi i}\int_{\operatorname{Re} q=\sigma} e^{qT}(q-r)^{-1}\mathcal{E}^-_q\left(\mathbb{1}_{(-\infty,h]}\right)(x)dq \\ &= 1 - \frac{e^{-rT}}{2\pi i}\int_{\operatorname{Re} q=\sigma} e^{qT}(q-r)^{-1}\mathcal{E}^-_q\left(\mathbb{1}_{(h,+\infty)}\right)(x)dq. \end{aligned}$$

The integral above is of almost the same form as the one in the formula for the no-touch digital option with the payoff function $G(x) \equiv 1$ and $\tilde{G}(q, 0+) = 1$, the only difference being the factor $(q-r)^{-1}$ in place of q^{-1}. Therefore, we obtain the asymptotic formulas (1.5)–(1.6) for the first-touch digitals in the same way as the corresponding formulas (1.4) and (1.3) for down-and-out options.

4. Leading term of asymptotics: Case $\nu \in (1, 2]$

4.1. **Wiener-Hopf factorization for an sRLPE: Case $\nu \in (1, 2]$.** For processes of order $\nu > 1$, we will use the following construction[6]. Let $q > 0$ be sufficiently large so that (C.8) holds for the chosen $[\omega_-, \omega_+]$, and $q^{1/\nu} > \max\{\lambda_+, -\lambda_-\}$. Define

$$\begin{aligned} \Psi(q,\eta) &= \frac{q+\psi(\eta)}{(q^{1/\nu}+id^{1/\nu}\eta)^{\nu_-}(q^{1/\nu}-id^{1/\nu}\eta)^{\nu_+}} \\ &= \frac{1+\psi(\eta)/q}{(1+i(d/q)^{1/\nu}\eta)^{\nu_-}(1-i(d/q)^{1/\nu}\eta)^{\nu_+}}, \end{aligned} \tag{4.1}$$

and set

$$I^\pm(q,\xi) = \pm\frac{1}{2\pi i}\int_{\operatorname{Im}\eta=\omega_\mp} \frac{\xi \ln\Psi(q,\eta)}{\eta(\xi-\eta)}d\eta, \tag{4.2}$$

$$\hat{I}^\pm(q,\xi) = \pm\frac{1}{2\pi i}\int_{\operatorname{Im}\eta=\omega_\mp} \frac{\ln\Psi(q,\eta)}{\eta-\xi}d\eta. \tag{4.3}$$

The integrals absolutely converge due to (2.19), and

$$I^\pm(q,\xi) = \hat{I}^\pm(q,0) - \hat{I}^\pm(q,\xi). \tag{4.4}$$

[6]This construction works for RLPEs of order $\nu > 1$ as well – see [11, Chapter 7]. In the case of sRLPEs, the proof simplifies.

It was proved in [11, Theorem 3.3] that

$$\phi^\pm_q(\xi) = (1 \mp i\xi\,(d/q)^{1/\bar\nu})^{-\nu_\pm} \exp I^\pm(q,\xi). \tag{4.5}$$

If σ is large enough, then, for any sRLPE, all the equalities above admit the analytic continuation w.r.t. q into the half-plane $\operatorname{Re} q \ge \sigma$, for ξ in the corresponding half-plane. However, this improved representation is especially useful for q in a sector $\Sigma_{\sigma,\theta}$, where $\theta > \pi/2$, if, for $q \in \Sigma_{\sigma,\theta}$ and $\pm \operatorname{Im}\xi \ge \pm\omega'_\mp$,

$$c(1+|\xi|/|q|^{1/\bar\nu})^{-\nu_\pm} \le |(1 \mp i\xi\,(d/q)^{1/\bar\nu})^{-\nu_\pm}| \le C(1+|\xi|/|q|^{1/\bar\nu})^{-\nu_\pm}, \tag{4.6}$$

where $C, c > 0$ are independent of q, ξ. Hence, it must be the case that $\nu > 1$, and θ must be in $(\pi/2, \pi\nu/2)$. In order that $I^\pm(q,\xi)$ and $\hat I^\pm(q,\xi)$ admit the analytic continuation w.r.t. q into $\Sigma_{\sigma,\theta}$, the θ may have to be smaller. The next lemma states that a θ with necessary and sufficient properties exists.

Lemma 4.1. *Let X be an sRLPE of order $\nu \in (1,2]$. Then there exist $\theta \in (\pi/2,\pi)$, $\sigma > 0$ and $\tilde\theta > 0$ such that*

(i) $q + \psi(\xi) \notin (-\infty, 0)$, $\quad \forall\, q \in \Sigma_{\sigma,\theta},\ \forall\, \xi \in \Sigma_{0,\tilde\theta} \cup (-\Sigma_{0,\tilde\theta})$;

(ii) $\operatorname{Re}(q^{1/\nu} \mp i d^{1/\nu}\xi) > 0$, $\quad \forall\, q \in \Sigma_{\sigma,\theta},\ \pm\operatorname{Im}\xi \ge \pm\omega'_\mp$;

(iii) (4.6) *holds;*

(iv) functions $I^\pm(q,\xi)$ and the Wiener-Hopf factors $\phi^\pm_q(\xi)$ admit the analytic continuation w.r.t. (q,ξ) into $\Sigma_{\sigma,\theta} \times \{\xi \mid \pm\operatorname{Im}\xi \ge \pm\omega'_\mp\}$, and satisfy the bounds

$$|I^\pm(q,\xi)| \le C, \tag{4.7}$$

$$c(1+|\xi|/|q|^{1/\nu})^{-\nu_\pm} \le |\phi^\pm_q(\xi)| \le C(1+|\xi|/|q|^{1/\nu}|)^{-\nu_\pm}, \tag{4.8}$$

where $c, C > 0$ are independent of $(q,\xi) \in \Sigma_{\sigma,\theta} \times \{\xi \mid \pm\operatorname{Im}\xi \ge \pm\omega'_\mp\}$;

(v) for any positive integer p, and any $\epsilon > 0$, the partial derivatives $\partial^p_q I^\pm(q,\xi)$ and $\partial^p_q \phi^\pm_q(\xi)$ admit upper bounds

$$|\partial^p_q I^\pm(q,\xi)| \ \le\ C_{p,\epsilon}|q|^{\epsilon-p}, \tag{4.9}$$

$$|\partial^p_q \phi^\pm_q(\xi)| \ \le\ C_{p,\epsilon}|q|^{\epsilon-p}(1+|\xi|/|q|^{1/\nu})^{-\nu_\pm}, \tag{4.10}$$

where $C_{p,\epsilon} > 0$ is independent of $(q,\xi) \in \Sigma_{\sigma,\theta} \times \{\xi \mid \pm\operatorname{Im}\xi \ge \pm\omega'_\mp\}$.

For the proof, see Appendix F.1.

4.2. **Convergence of the integral in** (3.10)**.** For $\nu \in (1,2]$, $\nu_+ \in (0,1]$, and it follows from (4.10) with sign "+" and (3.1) that for any $k \in \mathbb{Z}_+$, $\epsilon > 0$, and $a \in [0, \nu_+]$,

$$|\partial^k_q \phi^+_q(\xi)\hat G(\xi)| \le C_{k,\epsilon}(1+|\xi|)^{-1-a}|q|^{a/\nu-k+\epsilon}, \tag{4.11}$$

where $C_{k,\epsilon}$ is independent of $q \in \Sigma_{\sigma,\theta}$ and ξ in the half-plane $\operatorname{Im}\xi \ge \omega'_-$. Therefore, for k, a, ϵ satisfying the same conditions, function $\tilde G(q,\cdot) := \mathcal{E}^+_q G = \phi^+_q(D)G$ admits the bound

$$||\partial^k_q \tilde G(q,\cdot)||_{1/2+a+\epsilon} = O(|q|^{a/\nu-k+\epsilon}), \tag{4.12}$$

where $||\cdot||_s$ denotes the norm in $H^s(\mathbb{R})$. Letting $a = 0$ and using the Sobolev embedding theorem, we conclude that $\partial_q^k \tilde{G}(q,\cdot)$ is continuous on $(0,+\infty)$, and has the right limit at 0, which admits the bound

$$|\partial_q^k \tilde{G}(q,0+)| \le C_{k,\epsilon}|q|^{-k+\epsilon}, \tag{4.13}$$

for any $\epsilon > 0$, where $C_{k,\epsilon}$ is independent of $q \in \Sigma_{\sigma,\theta}$. Next, using general facts collected in Appendix A, we deduce from (4.12), that, for any $s \in (-1/2, 1/2)$ and $\epsilon > 0$, $\partial_q^k \mathbb{1}_{[0,+\infty)}\tilde{G}(q,\cdot) = O(|q|^{\epsilon-k})$ as an element of $\mathring{H}^s(\mathbb{R}_+)$.

Set $f_k(q,x) = \partial_q^k(q^{-1}\mathcal{E}_q^- \mathbb{1}_{[0,+\infty)}\tilde{G}(q,\cdot))$ (note that f_0 is f introduced in (3.12)) and denote by $\hat{f}_k(q,\xi)$ the Fourier transform of $f_k(q,x)$ w.r.t. x. Using (4.12) and applying (4.10) with sign "–", we conclude that, if $k \ge 1$, then, for any $\epsilon, \epsilon' > 0$, there exists $C_{\epsilon,\epsilon'}$ such that

$$|\hat{f}_k(q,\xi)| \le C_{\epsilon,\epsilon'}(1+|\xi|)^{-1-\nu_-+\epsilon'}|q|^{-2+\nu_-/\nu+\epsilon}. \tag{4.14}$$

Since $\nu_- \in (0,\nu)$, (4.14) implies that the integral (3.10) absolutely converges and defines a continuous function of T, with values in $\mathring{H}^{1/2+\nu_- -\epsilon'}(\mathbb{R}_+) \subset C^{\nu_- -2\epsilon'}(\mathbb{R}_+)$, for any $\epsilon' > 0$, by the Sobolev embedding theorem.

4.3. **Calculation of the asymptotics of the price.** *Step I.* We represent f_k as the sum $f_k^1 + f_k^2$, where f_k^j is the k-th derivative of j-term in (3.12) w.r.t. q, and denote by $\hat{f}_k^j(q,\xi)$ the Fourier transform of $f_k^j(q,x)$ w.r.t. x. The same argument as in §4.2 shows that $\hat{f}_k^2(q,\xi)$ obeys an estimate similar to but stronger than (4.14):

$$|\hat{f}_k^2(q,\xi)| \le C_{\epsilon,\epsilon'}(1+|\xi|)^{-2-\nu_-+\epsilon'}|q|^{-2+\nu_-/\nu+\epsilon}. \tag{4.15}$$

Hence, the inverse Laplace-Fourier transform (w.r.t. q and ξ, respectively) defines a continuous function of $T > 0$ with values in $C^{1+\nu_- -2\epsilon'}(\mathbb{R}_+)$, which can be included into the O-term of asymptotics. Thus, it remains to consider (3.10) with f_k^1 instead of f_k.

Step II. Take $s > 0$. It follows from (4.10) with sign "–" and (4.13), that, for any $\epsilon > 0$ and $m > 0$,

$$|\mathbb{1}_{|\xi|^s \le |q|}\hat{f}_k^1(q,\xi)| \le C_{\epsilon,\epsilon'}(1+|\xi|)^{-1-\nu_- -sm}|q|^{m-k+\nu_-/\nu+\epsilon} \tag{4.16}$$

(with different $C_{\epsilon,\epsilon'}$). Hence, if we take $k > m+1$ and substitute the inverse Fourier transform of $\mathbb{1}_{|\xi|^s \le |q|}\hat{f}_k^1(q,\xi)$ for $f_k^1(q,x)$ in (3.10), then we obtain a continuous function of T, with values in $\mathring{H}^{1/2+\nu_- +sm-\epsilon'}(\mathbb{R}_+) \subset C^{\nu_- +sm-2\epsilon'}(\mathbb{R}_+)$, for any $\epsilon' > 0$, which decays faster than x^{ν_-} as $x \downarrow 0$ (for calculation of the second term of asymptotics of the price and calculation of asymptotics of sensitivities, it may be important to take $m > 1$, hence, larger k). Hence, we can continue the calculation of the leading term replacing $f_k(q,x)$ in (3.10) with $f_k^{1,s}(q,x)$, the inverse Fourier transform of $\mathbb{1}_{|\xi|^s \ge |q|}\hat{f}_k^1(q,\xi)$.

Step III. We take $s \in (0, \nu)$, and, for a multi-index $p = (p_1, p_2, p_3)$, define

$$f^1(p,q,x) = q^{-1-p_1}(\partial_q^{p_3}\tilde{G}(q,0+))(\partial_q^{p_2}\mathcal{E}_q^-)(1+iD)^{-1}\delta)(x), \tag{4.17}$$

$$f^0(p,q,\xi) = q^{-1-p_1}\partial_q^{p_2}\Big((q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)]\Big)\,\partial_q^{p_3}\tilde{G}(q,0+)g_{\nu_-}(x), \tag{4.18}$$

where

$$g_{\nu_-}(x) = (1+iD)^{-\nu_- -1}\delta = (1/\Gamma(\nu_-+1))\mathbb{1}_{[0,+\infty)}(x)x^{\nu_-}e^{-x}. \tag{4.19}$$

Applying the Leibnitz rule to $f_k^1(q,x)$, we obtain a linear combinations of functions $f^1(p,q,x)$ with $|p| = k$. Using the argument at Step II, we can replace $f^1(p,q,x)$ with $f^{1,s}(p,q,x)$, the inverse Fourier transform of $\mathbb{1}_{|\xi|^s \ge |q|}\hat{f}^1(p,q,\xi)$.

Step IV. In Appendix F.2, we prove the following lemma.

Lemma 4.2. *Let X be an sRLPE of order $\nu \in (1,2)$, and let $\theta \in (\pi/2, \pi\nu/2)$ and $\sigma > 0$ be as in Lemma 4.1. Let $s \in (0,\nu)$. Then, for any $k \in \mathbb{Z}_+$, $q \in \Sigma_{\sigma,\theta}$, and ξ in the half-plane $\pm \operatorname{Im}\xi \ge \pm\omega_\mp$, such that $|q| \le |\xi|^s$,*

$$|\partial_q^k \hat{I}^\pm(q,\xi)| \le C_{s,k}|\xi|^{-1+s/\nu}|q|^{-k}, \tag{4.20}$$

where $C_{s,k}$ are independent of (q,ξ), and,

$$\phi_q^\pm(\xi) = (q/d)^{\nu_\pm/\nu}(1 \mp i\xi)^{-\nu_\pm}\exp[\hat{I}^\pm(q,0)] + R_\pm(q,\xi), \tag{4.21}$$

where $R_\pm(q,\xi)$ satisfies the following estimate: for any $\epsilon > 0$,

$$|\partial_q^k R_\pm(q,\xi)| \le C_{s,k,\epsilon}|q|^{\nu_\pm/\nu - k + \epsilon}|\xi|^{-1+s/\nu}, \tag{4.22}$$

where $C_{s,k,\epsilon}$ are independent of (q,ξ).

This lemma implies that, if s is sufficiently close to ν, then there exist $C > 0$, $s_1 > 1$ and $s_2 > \nu_-$, independent of $q \in \Sigma_{\sigma,\theta}$ and ξ in the half-plane $\operatorname{Im}\xi \le \omega'_+$, such that

$$|\widehat{f^{1,s}}(p,q,\xi) - \widehat{f^{0,s}}(p,q,\xi)| \le C|q|^{-s_1}(1+|\xi|)^{-1-s_2}. \tag{4.23}$$

Therefore, for any $\epsilon > 0$,

$$\int_{\operatorname{Re} q = \sigma} e^{qT}(f^{1,s}(p,q,x) - f^{0,s}(p,q,x))dq \in \mathring{H}^{1/2+s_2-\epsilon}(\mathbb{R}_+). \tag{4.24}$$

By the Sobolev embedding theorem, the integral (4.24) defines a function of class $C^{s_2-\epsilon}$, for any $\epsilon > 0$, which vanishes on $(-\infty, 0]$; hence, it tends to zero as $x \to 0$ faster than x^{ν_-}. It follows that the leading term of asymptotics does not change if we replace $f^{1,s}(p,q,x)$ with $f^{0,s}(p,q,x)$.

Step V. The same argument as at Step I shows that we can replace $f^{0,s}(p,q,x)$ with $f^0(p,q,x)$ (for details, see Appendix F.3). After that, we use (4.19) and derive the asymptotic formula (1.4) with

$$\kappa(T) = \frac{d^{-\nu_-/\nu}e^{-rT}}{2\pi i\,\Gamma(1+\nu_-)(-T)}\int_{\mathrm{Re}\,q=\sigma} e^{qT}\partial_q\left(q^{-1+\nu_-/\nu}\exp[\hat{I}^-(q,0)]\tilde{G}(q,0+)\right)dq. \tag{4.25}$$

We can deform the line of integration, integrate by parts back and obtain

$$\kappa(T) = \frac{d^{\nu_-/\nu}e^{-rT}}{2\pi i\,\Gamma(1+\nu_-)}\int_{\partial\Sigma_{\sigma,\theta}} e^{qT}q^{-1+\nu_-/\nu}\exp[\hat{I}^-(q,0)]\tilde{G}(q,0+)dq. \tag{4.26}$$

5. Two-term asymptotic formula: Case $\nu\in(0,1),\mu>0$

Our aim is to prove the following two-term asymptotic formula for the price

$$\begin{aligned} V(T,x) \quad &= \quad \left(1-\frac{d^0\sin(\gamma^0+\pi\nu/2)\Gamma(\nu-1)}{\pi\mu}x^{1-\nu}\right) \\ &\times\frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\,q=\sigma} e^{qT}\partial_q^k(q^{-1}e^{\hat{I}^-(q,0)}\tilde{G}(q,0+))dq + O(x^{\beta-\epsilon}). \end{aligned} \tag{5.1}$$

We start with necessary bounds for the Wiener-Hopf factors.

5.1. Bounds for $\phi_q^\pm(\xi)$.

If $\nu\in(0,1),\mu>0$, we have

$$\phi_q^-(\xi) = \exp[\hat{I}^-(q,0)-\hat{I}^-(q,\xi)], \tag{5.2}$$

where $\hat{I}^-(q,\xi)$ is given by

$$\hat{I}^-(q,\xi) \quad = \quad \frac{1}{2\pi i}\int_{\mathrm{Im}\,\eta=\omega_+}\frac{\ln\Psi(q,\eta)}{\eta-\xi}d\eta, \tag{5.3}$$

$$\Psi(q,\eta) \quad = \quad \frac{1+\psi(\eta)/q}{1-i\mu\eta/q} = \frac{q+\psi(\eta)}{q-i\mu\eta} \tag{5.4}$$

(see [11, Section 3.6.2]). If $\sigma>0$ is sufficiently large, then, for q in the half-plane $\mathrm{Re}\,q\ge\sigma$, and all η in the complex plane with the cuts $i(-\infty,\lambda_-]$ and $i[\lambda_+,+\infty)$, we have $q+\psi(\eta)\notin(-\infty,0]$. Hence, we can transform the line of integration in (4.3) to the cut $i[\lambda_+,+\infty)$, and obtain

$$\hat{I}^-(q,\xi) = \frac{1}{2\pi i}\int_{\lambda_+}^{+\infty}\frac{\ln\Phi(q,z)}{z+i\xi}dz, \tag{5.5}$$

where

$$\Phi(q,z) \quad = \quad \frac{q+\mu z+\psi^0(iz-0)}{q+\mu z+\psi^0(iz+0)} = 1+\frac{\psi^0(iz-0)-\psi^0(iz+0)}{q+\mu z+\psi^0(iz+0)} \tag{5.6}$$

As $z \to +\infty$,

$$\Phi(q, z) = 1 + \frac{d_-^0 e^{-i\pi\nu/2} - d_+^0 e^{i\pi\nu/2}}{\mu} z^{\nu-1} + O(z^{-\beta}), \tag{5.7}$$

where $\beta = \min\{1, 2(1-\nu)\}$, therefore

$$\hat{I}^-(q, \xi) = \hat{I}_0^-(\xi) + R_-(q, \xi), \tag{5.8}$$

where

$$\hat{I}_0^-(\xi) = -\Omega_- \int_{\lambda_+}^{+\infty} \frac{z^{\nu-1}}{z + i\xi} dz, \tag{5.9}$$

$$\Omega_- = -\frac{d_-^0 e^{-i\pi\nu/2} - d_+^0 e^{i\pi\nu/2}}{2\pi\mu i} = \frac{d^0 \sin(\gamma^0 + \pi\nu/2)}{\pi\mu}, \tag{5.10}$$

$\gamma^0 = \arg d_+^0$, $d_0 = |d_\pm^0|$, and $R_-(q, \xi)$ admits estimates

$$|R_-(q, \xi)| \le C \int_{\lambda_+}^{+\infty} \frac{z^{-\beta} dz}{|z| + |\xi|} \le C_1 (1 + |\xi|)^{-\beta} \tag{5.11}$$

(if $\beta = 1$, then an additional factor $\ln(1 + |\xi|)$ needs to be added; for simplicity, below, we replace $\beta = 1$ with $\beta \in (1-\nu, 1)$). To obtain bounds for the derivatives of $R_-(q, \xi)$, we use

$$\partial_q^p \ln \Phi(q, z) = (-1)^{p-1}[(q+\mu z+\psi^0(iz-0))^{-p} - (q+\mu z+\psi^0(iz+0))^{-p}], \quad p \ge 1, \tag{5.12}$$

which yields the global bound

$$|\partial_q^p \ln \Phi(q, z)| \le C(1 + |z|)^\nu (|q| + |z|)^{-p-1}, \tag{5.13}$$

and, as a corollary,

$$|\partial_q^p R_-(q, \xi)| \le C_p \int_{\lambda_+}^{+\infty} \frac{(1 + |z|)^\nu dz}{(|q| + |z|)^{p+1}(|\xi| + |z|)^2} \le C_{1p} |q|^{\nu-p} (|q| + |\xi|)^{-2}. \tag{5.14}$$

A bound for $\hat{I}_0^-(\xi)$ is immediate from (5.9):

$$|\hat{I}_0^-(\xi)| \le C(1 + |\xi|)^{\nu-1}, \tag{5.15}$$

and, finally, we derive the bounds for $\phi_q^-(\xi)$ and its derivatives:

$$|\phi_q^-(\xi)| \le C, \tag{5.16}$$

$$|\partial_q^p \phi_q^-(\xi)| \le C|q|^{1-p}(|q| + |\xi|)^{-2}, \quad p \ge 1. \tag{5.17}$$

Notice that the bounds above are valid for q in the half-plane $\operatorname{Re} q \ge \sigma$ and ξ in the half-plane $\operatorname{Im} \xi \ge \omega'_-$.

For $\phi^+_q(\xi)$, we use the Wiener-Hopf factorization formula:

$$\phi^+_q(\xi) = \frac{q}{q - i\mu\xi + \psi^0(\xi)} \frac{1}{\phi^-_q(\xi)},$$

which is valid in the strip $\mathrm{Im}\,\xi \in [\omega'_-, \omega'_+]$, and, therefore, derive estimates, for $p \in \mathbb{Z}$,

$$|\partial^p_q \phi^+_q(\xi)| \le C|q|/(|q - i\mu\xi| + |\xi|^\nu)^{1+p} \tag{5.18}$$

for ξ in this strip (and q in the half-plane $\mathrm{Re}\,q \ge \sigma$).

It follows from (3.1) and (5.18) that, for any $\epsilon > 0$ and $p \in \mathbb{Z}_+$,

$$||\partial^p_q \tilde{G}(q, x)||_{1/2-\epsilon} \le C_p|q|^{1-p}. \tag{5.19}$$

5.2. **Simplification: the first step.** We start to distill the leading term of $\mathbb{1}_{(0,+\infty)}\tilde{G}(q, x)$ in (3.10) using (B.1)

$$\begin{aligned} \mathbb{1}_{(0,+\infty)}\tilde{G}(q, x) &= \tilde{G}(q, 0+)(1 + iD)^{-1}\delta \\ &\quad +[\tilde{G}(q, 0+)(+0) + \tilde{G}_x(q, 0+)](1 + iD)^{-2}\delta \\ &\quad +(1 + iD)^{-2}\mathbb{1}_{(0,+\infty)}(x)(1 + iD)^2\chi(x)\tilde{G}(q, x) \\ &\quad +(1 - \chi(x))\tilde{G}(q, x), \end{aligned} \tag{5.20}$$

where $\chi \in C^\infty(\mathbb{R})$, $\chi(x) = 1, x < \ln K/3$, $\chi(x) = 0, x > \ln K/2$. Denote by $GG_j(q, x)$, $j = 1, 2, 3, 4$, the terms on the RHS of (5.20), and by $V_j(T, x)$ the function obtained replacing $\mathbb{1}_{(h,+\infty)}\mathcal{E}^+_q G(q, x) = \mathbb{1}_{(0,+\infty)}\tilde{G}(q, x)$ in (3.10) with GG_j. Then

$$V(T, x) = \sum_{j=1}^{4} V_j(T, x). \tag{5.21}$$

On the strength of the properties of the EPV-operator $\mathcal{E}^-_q$, $V_4(T, x) = 0$ for $x < \ln K/2$, hence, $V_4(T, x)$ has no impact on the leading term of asymptotics of $V(T, x)$ as $x \downarrow 0$.

Next, for the standard payoffs and model classes of processes, $\chi\tilde{G}(q, \cdot) \in C^\infty(\mathbb{R})$. Moreover, for any $p, m \in \mathbb{Z}$,

$$||\partial^p_q \chi\tilde{G}(q, \cdot)||_m \le C_{p,m}|q|^{1-p}. \tag{5.22}$$

It follows that, for any $\epsilon > 0$, $|q|^{p-1}\mathbb{1}_{(0,+\infty)}\partial^p_q\chi\tilde{G}(q, \cdot) \in \mathring{H}^{1/2-\epsilon}(\mathbb{R})$ uniformly in q in the half-plane $\mathrm{Re}\,q \ge \sigma$, and, on the strength of (5.16) and (5.17),

$$|q|^{1-p}\partial^p_q GG_3(q, \cdot) \in \mathring{H}^{5/2-\epsilon}(\mathbb{R}_+)$$

uniformly in q in the half-plane $\mathrm{Re}\,q \ge \sigma$. It follows that if $k \ge 2$, then $V_3(T, \cdot) \in \mathring{H}^{5/2-\epsilon}(\mathbb{R}_+)$. Hence, for any $\epsilon > 0$, $V_3(T, x) = O(x^{2-\epsilon})$ as $x \downarrow 0$.

Next, since $(1+iD)^{-2}\delta \in \mathring{H}^{3/2-\epsilon}(\mathbb{R}_+)$ for any $\epsilon > 0$, a similar argument shows that, for any $\epsilon > 0$, $V_2(T, x) = O(x^{1-\epsilon})$ as $x \downarrow 0$. Thus, it remains to consider $V_1(T, x)$.

5.3. **Second step: Study of $V_1(T,x)$.** From (5.2), (5.8), (5.11), (5.14) and (5.15),

$$\phi_q^-(\xi)(1+i\xi)^{-1} = \exp[\hat{I}^-(q,0)](1+i\xi)^{-1}(1-\hat{I}_0^-(\xi)) + R_{--}(q,\xi), \tag{5.23}$$

where $R_{--}(q,\xi)$ and its derivatives w.r.t. q admit estimates of the same form as $R_-(q,\xi)$ and its derivatives w.r.t. q but with an additional factor $(1+|\xi|)^{-1}$ on the RHS of bounds (5.11), (5.14). It follows that if we omit the last term on the RHS of (5.23), then $V_1(T,x)$ will change by the term of order $O(x^{\beta-\epsilon})$, for any $\epsilon>0$.

So far, we have shown that if, in (3.10), we replace $\left(\mathcal{E}_q^-\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+G\right)(x)$ with

$$\tilde{G}(q,0+)\exp[\hat{I}^-(q,0)]((1+iD)^{-1}\delta - \hat{I}_0^-(D)(1+iD)^{-1}\delta),$$

the option price will change by $O(x^{1-\epsilon})$, for any $\epsilon>0$. The first term gives the leading term of asymptotics; since $(1+iD)^{-1}\delta(x) = e^{-x}\mathbb{1}_{(0,+\infty)}(x)$, the leading term is a constant:

$$\kappa(T) = \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{qT}\partial_q^k\left(q^{-1}e^{\hat{I}^-(q,0)}\tilde{G}(q,0+)\right)dq, \tag{5.24}$$

and the next term of asymptotics equals

$$V_{01}(T,x) = F(x)\frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{qT}\partial_q^k\left(q^{-1}e^{\hat{I}^-(q,0)}\tilde{G}(q,0+)\right)dq, \tag{5.25}$$

where

$$F(x) = -\frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+} e^{ix\xi}\hat{I}_0^-(\xi)(1+i\xi)^{-1}d\xi. \tag{5.26}$$

Since the integral in (5.9) absolutely converges, and

$$\int_{\lambda_+}^{+\infty}\left|\frac{z^{\nu-1}}{z+i\xi}\right|dz \le C(1+|\xi|)^{\nu-1},$$

we can substitute (5.9) into (5.26) and apply the Fubini theorem

$$F(x) \;=\; \Omega_-\int_{\lambda_+}^{+\infty}dz\,z^{\nu-1}\frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+}d\xi\,e^{ix\xi}(z+i\xi)^{-1}(1+i\xi)^{-1} \tag{5.27}$$

Rescaling if necessary, we may assume that $1<\lambda_+$. Then

$$(z+i\xi)^{-1}(1+i\xi)^{-1} = (z-1)^{-1}((1+i\xi)^{-1}-(z+i\xi)^{-1}),$$

and, therefore,

$$\frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+}d\xi\,e^{ix\xi}(z+i\xi)^{-1}(1+i\xi)^{-1} = (z-1)^{-1}(e^{-x}-e^{-zx})\mathbb{1}_{(0,+\infty)}(x),$$

and

$$F(x) \;=\; \Omega_-\int_{\lambda_+}^{+\infty}\frac{z^{\nu-1}}{z-1}(e^{-x}-e^{-zx})\mathbb{1}_{(0,+\infty)}(x)dz. \tag{5.28}$$

As $x \downarrow 0$, the integrand decreases to 0 pointwise, and it is uniformly bounded by $z^{\nu-1}/(z-1)$, which is integrable, since $\nu \in (0,1)$. Hence, $F(0+) = 0$. Similarly, we can differentiate under the integral sign w.r.t. $x > 0$ to get

$$\begin{aligned} F'(x)/\Omega_- &= \int_{\lambda_+}^{+\infty} z^{\nu-1}\frac{ze^{-zx}-e^{-x}}{z-1}dz \\ &= \int_0^{+\infty} z^{\nu-1}e^{-zx}dz - \int_0^{\lambda_+} z^{\nu-1}e^{-zx}dz + \int_{\lambda_+}^{+\infty} z^{\nu-1}\frac{e^{-zx}-e^{-x}}{z-1}dz. \end{aligned}$$

Since $\nu \in (0,1)$, the second and third terms are uniformly bounded w.r.t. $x > 0$, and we obtain

$$F'(x) = \Omega_-\Gamma(\nu)x^{-\nu} + O(1).$$

Integrating, we find

$$F(x) = -\Omega_-\Gamma(\nu-1)x^{1-\nu} + O(x), \tag{5.29}$$

and, finally, arrive at (5.1).

5.4. **A more accurate second term of asymptotics and the next terms.** If we do not resort to the simplification (5.8), we can use (5.5) directly, and, instead of (5.25), calculate the second term of asymptotics as

$$V_{01}(T,x) = \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q^k\left(q^{-1}e^{\hat{I}^-(q,0)}\tilde{G}(q,0+)F(q,x)\right)dq, \tag{5.30}$$

where

$$F(q,x) = \frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega'_+} e^{ix\xi}(1-\exp[\hat{I}^-(q,\xi)])(1+i\xi)^{-1}d\xi. \tag{5.31}$$

In this case, the remainder is of order $O(x^{1-\epsilon})$, for any $\epsilon > 0$, rather than of order $O(x^{\beta-\epsilon})$. Furthermore, it can be shown that if, in (5.30), we replace $F(q,x)$ with $F(1,x)$, the final result will change by $O(x^{1-\epsilon})$, for any $\epsilon > 0$. If $2(1-\nu) \geq 1$, this gives no real improvement but if $2(1-\nu) < 1$, then one can use the two-term asymptotic expansion

$$1 - \exp[\hat{I}^-(1,\xi)] = -\hat{I}^-(1,\xi) - \hat{I}^-(1,\xi)^2/2 + O(|\xi|^{3(1-\nu)})$$

(or even more than two-term expansion, if $3(1-\nu) < 1$). The same calculation as the one leading to (5.28) now gives

$$F(1,x) = -\frac{1}{2\pi i}\int_{\lambda_+}^{+\infty}\frac{\ln\Phi(1,z)}{z-1}(e^{-x}-e^{-zx})\mathbb{1}_{(0,+\infty)}(x)dz + O(x^{2(1-\nu)}), \tag{5.32}$$

and, should one wish it, the next term of the form

$$F_2(1,x) = \frac{1}{8\pi^2}\int_{\lambda_+}^{+\infty} dz\,\ln\Phi(1,z)\int_{\lambda_+}^{+\infty} dz'\,\ln\Phi(1,z') \times\frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega'_+} d\xi\, e^{ix\xi}(z+i\xi)^{-1}(z'+i\xi)^{-1}(1+i\xi)^{-1} \tag{5.33}$$

added (and reduced to the double integral w.r.t. $dz\,dz'$), instead of the $O(x^{2(1-\nu)})$ in (5.32); then the remainder in (5.32) will be of order $O(x^{3(1-\nu)})$. Both expressions in (5.32) and (5.33) can be evaluated up to errors of order $O(x^{1-\epsilon})$ using the asymptotic expansion of $\Phi(1,z)$ as $z\to+\infty$.

6. Two-term asymptotic formula: Case $\nu\in(1,2), \mu\neq 0$

6.1. **Reduction to the case $\nu_-=0$.** We start with formula (3.10), and introduce $\widetilde{V}(T,x)=(1+iD)^{\nu_-}V(T,x)$ and $\widetilde{\phi}^-_q(\xi)=(1+i\xi)^{\nu_-}\phi^-_q(\xi)$. Then

$$\widetilde{V}(T,x)=\frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\,q=\sigma} e^{qT}\partial_q^k\left(q^{-1}\widetilde{\phi}^-_q(D)\mathbb{1}_{(0,+\infty)}\tilde{G}(q,x)\right)dq. \tag{6.1}$$

The idea is to calculate the asymptotics of $\widetilde{V}(T,x)$ as we calculated the asymptotics of $V(T,x)$ in the case $\nu\in(0,1),\mu>0$, with certain modifications, and derive a two-term asymptotic formula of the form

$$\widetilde{V}(T,x)=\widetilde{\kappa}(T)+\widetilde{\kappa}_1(T)x^{\nu-1}+O(x^{\nu-1+\beta-\epsilon}),\quad x\downarrow 0, \tag{6.2}$$

where $\beta=\min\{1,2(\nu-1)\}$, $\epsilon>0$ is arbitrary, and

$$\widetilde{\kappa}(T) = \frac{e^{-rT}d^{-\nu_-/\nu}}{2\pi i(-T)^k}\int_{\mathrm{Re}\,q=\sigma} e^{qT}\partial_q^k\left(q^{-1+\nu_-/\nu}e^{\hat{I}^-(q,0)}\tilde{G}(q,x)\right)dq, \tag{6.3}$$

$$\widetilde{\kappa}_1(T) = \widetilde{\kappa}(T)\frac{-\Gamma(1-\nu)\mu\sin(\gamma+\pi\nu/2)}{\pi d}. \tag{6.4}$$

Assuming that (6.2) has been proved, we rewrite (6.2) in the form

$$\widetilde{V}(T,x)=\widetilde{\kappa}(T)e^{-x}\mathbb{1}_{(0,+\infty)}(x)+\widetilde{\kappa}_1(T)x^{\nu-1}e^{-x}\mathbb{1}_{(0,+\infty)}(x)+O(x^{\nu-1+\beta-\epsilon}e^{-x}\mathbb{1}_{(0,+\infty)}(x)),$$

apply $(1+iD)^{-\nu_-}$, and obtain

$$\begin{aligned} V(T,x) &= \widetilde{\kappa}(T)(1+iD)^{-\nu_-}e^{-x}\mathbb{1}_{(0,+\infty)}(x)\\ &\quad+\widetilde{\kappa}_1(T)(1+iD)^{-\nu_-}x^{\nu-1}e^{-x}\mathbb{1}_{(0,+\infty)}(x)\\ &\quad+(1+iD)^{-\nu_-}O(x^{\nu-1+\beta-\epsilon}e^{-x}\mathbb{1}_{(0,+\infty)}(x)). \end{aligned} \tag{6.5}$$

Since $(1+iD)^{-\nu_-}$ is an integral operator with the non-negative kernel

$$\Gamma(\nu_-)^{-1}|x|^{\nu_--1}e^{x}\mathbb{1}_{(-\infty,0)}(x),$$

we can interchange the O-sign and $(1+iD)^{-\nu_-}$ in the last term in the RHS of (6.5). Further, for $a>0$,

$$\mathcal{F}_{x\to\xi}\left(x^{a-1}e^{-x}\mathbb{1}_{(0,\infty)}(x)\right)=\Gamma(a)(1+i\xi)^{-a}, \tag{6.6}$$

where $\mathcal{F}_{x\to\xi}$ denotes the Fourier transform, therefore,

$$(1+iD)^{-\nu_-}x^{a-1}e^{-x}\mathbb{1}_{(0,\infty)}(x)=\frac{\Gamma(a)}{\Gamma(a+\nu_-)}x^{a+\nu_--1}e^{-x}\mathbb{1}_{(0,\infty)}(x). \tag{6.7}$$

Applying (6.7) to (6.5), we derive

$$V(T,x)=\frac{\widetilde{\kappa}(T)\Gamma(1)}{\Gamma(1+\nu_-)}x^{\nu_-}+\frac{\widetilde{\kappa}_1(T)\Gamma(\nu)}{\Gamma(\nu+\nu_-)}x^{\nu_-+\nu-1}+O(x^{\nu_-+\nu-1+\beta-\epsilon}). \tag{6.8}$$

Using (6.4), we simplify (6.8)

$$\begin{aligned} V(T,x) &= \frac{\widetilde{\kappa}(T)}{\Gamma(1+\nu_-)}x^{\nu_-} \\ &\quad\times\left(1-\frac{\mu\sin(\gamma+\pi\nu/2)\Gamma(1-\nu)\Gamma(\nu)\Gamma(1+\nu_-)}{\Gamma(\nu+\nu_-)\pi d}x^{\nu-1}\right) \\ &\quad+O(x^{\nu_-+\nu-1+\beta-\epsilon}) \end{aligned} \tag{6.9}$$

6.2. **Proof of** (6.2)**.** We start with (6.1). Evidently, $\widetilde{\phi_q^-}(\xi)$ admits estimates of the same form as $\phi_q^-(\xi)$, with an additional factor $(1+|\xi|)^{\nu_-}$ on the RHSs, and the same is true of the derivatives of $\widetilde{\phi_q^-}(\xi)$. Therefore, a straightforward modification of the constructions used to calculate the two-term expansions in the case $\nu\in(0,1),\mu>0$ (we must add the factor $|q|^{\nu_-/\nu+\epsilon}$ in the estimates for the minus-factor, and replace the factor $|q|$ with the factor $|q|^{\nu_+/\nu+\epsilon}$ in the estimates for the plus-factor, where $\epsilon>0$ is arbitrary) now gives

$$\begin{aligned} \widetilde{V}(T,x) &= \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re}q=\sigma}e^{qT}\partial_q^k\left(q^{-1}\tilde{G}(q,0+)\widetilde{\phi_q^-}(D)(1+iD)^{-1}\delta(x)\right)dq \\ &\quad+O(x^{1-\epsilon}),\quad \forall\ \epsilon>0. \end{aligned} \tag{6.10}$$

We represent

$$\begin{aligned} \widetilde{\phi_q^-}(D)(1+iD)^{-1}\delta(x) &= (1+iD)^{\nu_--1}(1+iD(d/q)^{1/\nu})^{-\nu_-} \\ &\quad\times\exp[\hat{I}^-(q,0)-\hat{I}^-(q,D)]\delta(x), \end{aligned}$$

where $\hat{I}^-(q,\xi)$ is given by (4.3), in the form

$$\begin{aligned} \widetilde{\phi_q^-}(D)(1+iD)^{-1}\delta(x) &= (q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)](1+iD)^{-1}\delta(x) \\ &\quad+(q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)](1+iD)^{-1}(1-\exp[\hat{I}^-(q,D)])\delta(x) \\ &\quad+(q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)-\hat{I}^-(q,D)](1+iD)^{-1} \\ &\quad\times\left[(1+iD)^{\nu_-}((q/d)^{1/\nu}+iD))^{-\nu_-}-1\right]\delta(x) \end{aligned} \tag{6.11}$$

Function

$$f_3(q,\xi) = (q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)-\hat{I}^-(q,\xi)](1+i\xi)^{-1}\left[(1+i\xi)^{\nu_-}((q/d)^{1/\nu}+i\xi))^{-\nu_-}-1\right]$$

admits the global bound via $C|q|^{\nu_-/\nu}(1+|\xi|)^{-1}$, and its derivatives w.r.t. q admit the bound

$$|\partial_q^p f_3(q,\xi)| \le C_{q,\epsilon}|q|^{\nu_-/\nu-p+\epsilon}(1+|\xi|)^{-1},$$

for any $\epsilon>0$. On the set $|q|\le|\xi|^\nu$, $f_3(q,\xi)$ admits the bound via $C|q|^{\nu_-/\nu+1}(1+|\xi|)^{-2}$, and its derivatives w.r.t. q admit the bound

$$|\partial_q^p f_3(q,\xi)| \le C_{q,\epsilon}|q|^{\nu_-/\nu+1-p+\epsilon}(1+|\xi|)^{-2},$$

for any $\epsilon>0$. Denote by $\widetilde{V}_3(T,x)$ the function defined by (6.10) with $f_3(q,D)$ substituted for $\widetilde{\phi}_q^-(D)(1+iD)^{-1}$. It follows from the bounds, which we just obtained, that $\widetilde{V}_3(T,x)=O(x^{1-\epsilon})$, for any $\epsilon>0$. Thus, we can disregard the last term in (6.11).

Denote by $\widetilde{V}_0(T,x)$ the function defined by (6.10) with $(q/d)^{\nu_-/\nu}\exp[\hat{I}^-(q,0)](1+iD)^{-1}\delta(x)$ substituted for $\widetilde{\phi}_q^-(D)(1+iD)^{-1}$. This is the leading term of asymptotics:

$$\widetilde{V}_0(T,x)=\widetilde{\kappa}(T)e^{-x}\mathbb{1}_{(0,+\infty)}(x).$$

Similarly, since $\hat{I}^-(q,\xi)=O(|\xi|^{1-\nu})$, and the estimates for the derivatives w.r.t. q are even better: additional negative powers of $|q|$ appear, the second term on the RHS of (6.11) gives rise to the second term of asymptotics, of order $O(x^{1-\nu})$[7]. Furthermore, if we replace $1-\exp[\hat{I}^-(q,D)]$ with $-\hat{I}^-(q,D)$, then an additional error term in the asymptotic formula will be of order $O(x^{2(1-\nu)-\epsilon})$, for any $\epsilon>0$. Thus, for any $\epsilon>0$,

$$\widetilde{V}(T,x)=\widetilde{V}_0(T,x)+\widetilde{V}_1(T,x)+O(x^{\beta-\epsilon}),$$

where $\beta=\min\{2(\nu-1),1\}$, and

$$\widetilde{V}_1(T,x)=\frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re}q=\sigma}e^{qT}\partial_q^k\left(q^{-1}\tilde{G}(q,0+)(q/d)^{\nu_-/\nu}e^{\hat{I}^-(q,0)}F(q,x)\right)dq, \tag{6.12}$$

where

$$\begin{aligned} F(q,x) &= -(1+iD)^{-1}\hat{I}^-(q,D)\delta(x) \\ &= -\frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+}e^{ix\xi}(1+i\xi)^{-1}\hat{I}^-(q,\xi)d\xi, \end{aligned} \tag{6.13}$$

and

$$\hat{I}^-(q,\xi)=-\frac{1}{2\pi i}\int_{\operatorname{Im}\eta=\omega_+}\frac{\ln\Psi(q,\eta)}{\eta-\xi}d\eta. \tag{6.14}$$

[7]If $\nu-1<1/2$, then, similarly to the case $\nu\in(0,1),\mu>0$, one can derive a rather complicated third term as well, of order $O(x^{2(1-\nu)})$

Assuming that X is an sRLPE, we can transform the contour of integration into $\mathcal{L}_{\omega_+,\tilde\theta} = \{z = i\omega_+ + \rho e^{i\phi} \mid \rho \ge 0, \phi = \tilde\theta \text{ or } \phi = \pi - \tilde\theta\}$; then the integral becomes an absolutely convergent one, and since

$$|\hat I^-(q,\xi)| \le \frac{1}{2\pi i}\int_{\mathcal{L}_{\omega_+,\tilde\theta}} \left|\frac{\ln\Psi(q,\eta)}{\eta-\xi}\right| d\eta \le C(1+|\xi|)^{1-\nu},$$

we can substitute (6.14) into (6.13) and change the order of integration to obtain

$$F(q,x) = -\frac{1}{2\pi}\int_{\mathcal{L}_{\omega_+,\tilde\theta}} d\eta \, \ln\Psi(q,\eta) \times \frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+} d\xi \, e^{ix\xi}(1+i\xi)^{-1}(-i\eta+i\xi)^{-1}. \tag{6.15}$$

Define $\widetilde V_{10}(T,x)$ by (6.12) with $F(1,x)$ instead of $F(q,x)$.

Lemma 6.1. *For any $\epsilon > 0$, $\widetilde V_1(T,x) - \widetilde V_{10}(T,x) = O(x^{1-\epsilon})$.*

Proof. It suffices to prove that for any $p \in \mathbb{Z}$ and $\epsilon > 0$,

$$|\partial_q^p \hat I^-(q,\xi)| \le C_{p,\epsilon}|q|^{1/\nu-p}(1+|\xi|)^{-1+\epsilon}. \tag{6.16}$$

If $C > 0$ is sufficiently large, and $|\eta| \ge C|q|^{1/\nu}$, then

$$|\Psi(q,\eta)/\Psi(1,\eta) - 1| \le C_1|q|^{1/\nu}(1+|\eta|)^{-1},$$

where C_1 is independent of these (q,η), and on $|\eta| \le C|q|^{1/\nu}$,

$$|\Psi(q,\eta)/\Psi(1,\eta)| \le C_1|q|.$$

The two estimates just obtained and integration in (6.14) over the deformed contour $\mathcal{L}_{\omega_+,\tilde\theta}$. give (6.16) with $p = 0$. If $p > 0$, we use

$$|\partial_q^p \ln\Psi(q,\eta)| \le C_p|q|^{1/\nu-p}(|q|^{1/\nu}+|\eta|)^{-1}.$$

□

6.3. **Calculation of the second term.** Thus, it remains to calculate the leading term of asymptotics of $\widetilde V_{10}(T,x)$. Rescaling if necessary, we may assume that $1 < \omega_+$. Then we calculate the inner integral in (6.15) (multiplied by $1/(2\pi)$) as

$$(1+i\eta)^{-1}(e^{ix\eta} - e^{-x}),$$

and obtain

$$F(1,x) = \frac{1}{2\pi}\int_{\mathcal{L}_{\omega_+,\tilde\theta}} \frac{\ln\Psi(1,\eta)(e^{ix\eta}-e^{-x})}{1+i\eta} d\eta. \tag{6.17}$$

Recall that $\ln\Psi(1,\eta) = O(|\eta|^{-\beta})$, where $\beta = \min\{1, \nu-1\}$. Hence, in (6.17), the absolute value of the integrand is bounded by an integrable function, which is independent of x, and converges to 0 point-wise as $x \downarrow 0$. We conclude that $F(1, 0+) = 0$.

A similar argument shows that we can differentiate under the integral sign in (6.17) and obtain

$$\begin{aligned} F_x(1,x) &= -\frac{1}{2\pi}\int_{\mathcal{L}_{\omega_+,\tilde\theta}} \frac{\ln\Psi(1,\eta)(i\eta e^{ix\eta}+e^{-x})}{1+i\eta}d\eta \\ &= -\frac{1}{2\pi}\int_{\mathcal{L}_{\omega_+,\tilde\theta}} \ln\Psi(1,\eta)e^{ix\eta}d\eta + O(1). \end{aligned} \tag{6.18}$$

We deform the contour of integration in (6.18) further and integrate over the cut in the upper half-plane

$$F_x(1,x) = \frac{1}{2\pi i}\int_{\lambda_+}^{+\infty} \ln\frac{\Psi(1,iz-0)}{\Psi(1,iz+0)}e^{-zx}dz + O(1). \tag{6.19}$$

We have

$$\ln\frac{\Psi(1,iz-0)}{\Psi(1,iz+0)} = \frac{\mu z}{\psi^0(iz-0)} - \frac{\mu z}{\psi^0(iz+0)} + O(z^{\nu-1-\beta}).$$

Since $\nu > 1$, $d^0_\pm = d_\pm$, and, therefore, as $z \to +\infty$,

$$\psi^0(iz\pm 0) = d^0_\pm e^{\pm\pi\nu/2}z^\nu(1+O(z^{-\beta}) = de^{\pm(\gamma+\pi\nu/2)}z^\nu(1+O(z^{-\beta})),$$

we obtain

$$\ln\frac{\Psi(1,iz-0)}{\Psi(1,iz+0)} = \frac{\mu\sin(\gamma+\pi\nu/2)}{d}z^{1-\nu}(1+O(z^{-\beta})).$$

Substituting into (6.19) (and, if $3(1-\nu) > -1$, making an additional asymptotic expansion in the O-term), we find

$$F_x(1,x) = \frac{\mu\sin(\gamma+\pi\nu/2)\Gamma(2-\nu)}{\pi d}x^{\nu-2} + O(1),$$

and then, integrating,

$$F(1,x) = -\frac{\mu\sin(\gamma+\pi\nu/2)\Gamma(1-\nu)}{\pi d}x^{\nu-1} + O(x).$$

Finally, substituting into $F(1,x)$ into (6.12) in place of $F(q,x)$, we obtain

$$\widetilde{V}_{10}(T,x) = \widetilde{\kappa}_1(T)x^{\nu-1} + O(x),$$

where $\widetilde{\kappa}_1(T)$ is given by (6.4), which finishes the proof of the two-term asymptotic formulas (6.2) and (6.9).

7. Two-term asymptotic formula: Case $\nu \in (0,1), \mu < 0$

In this case, $\nu_- = 1$. As in the case $\nu \in (1,2)$, the first step is the reduction to the case $\nu_- = 0$. After that, the calculations are essentially the same as in the case $\nu \in (0,1), \mu > 0$, only this time we will have to use the Wiener-Hopf factorization and estimates for $\phi^+_q(\xi)$ in order to derive the estimates for $\phi^-_q(\xi)$, which we need.

7.1. **Reduction to the case $\nu_- = 0$.** We start with formula (3.10), and introduce $\widetilde{V}(T,x) = (1+iD)^{\nu_-}V(T,x)$. Then (6.1) holds, with $\widetilde{\phi_q^-}(\xi) = (1+i\xi)\phi_q^-(\xi)$. In the following subsections, we will prove that

$$\widetilde{V}(T,x) = \widetilde{\kappa}(T) + \widetilde{\kappa}_1(T)x^{1-\nu} + O(x^{1-\nu+\beta-\epsilon}), \quad x \downarrow 0, \tag{7.1}$$

where $\beta = \min\{1, 2(1-\nu)\}$, $\epsilon > 0$ is arbitrary, and

$$\widetilde{\kappa}(T) = \frac{e^{-rT}}{2\pi i(-T)^k(-\mu)} \int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q^k \left(e^{\hat{I}^-(q,0)}\tilde{G}(q,0+)\right) dq, \tag{7.2}$$

$$\widetilde{\kappa}_1(T) = \widetilde{\kappa}(T)\frac{-\Gamma(\nu-1)d^0\sin(\gamma^0+\pi\nu/2)}{\pi\mu}. \tag{7.3}$$

Assuming that (7.1) has been proved, we find $V(T,x)$ integrating the ordinary differential equation $V_x(T,x) + V(T,x) = \widetilde{V}(T,x)$ subject to $V(T,0) = 0$, with parameter T, and derive

$$V(T,x) = \widetilde{\kappa}(T)x\left(1 + \frac{d^0\sin(\gamma^0+\pi\nu/2)\Gamma(\nu-2)}{\pi\mu}x^{1-\nu}\right) + O(x^{1-\nu+\beta-\epsilon}). \tag{7.4}$$

7.2. **Bounds for $\phi_q^{\pm}(\xi)$.** The calculations in this section are mirror reflections of the calculations in §5.1. We start with

$$\phi_q^+(\xi) = \exp[\hat{I}^+(q,0) - \hat{I}^+(q,\xi)], \tag{7.5}$$

where

$$\hat{I}^+(q,\xi) = \frac{1}{2\pi i}\int_{-\infty}^{\lambda_-} \frac{\ln\Phi(q,z)}{z+i\xi}dz, \tag{7.6}$$

and $\Phi(q,z)$ is given by (5.6). However, only the resulting bounds but not asymptotic expansions obtained in this way are useful here: indeed, we will get the asymptotic expansion of $\phi_q^+(\xi)$ for ξ in the half-plane $\mathrm{Im}\,\xi \ge \omega'_-$, whereas we need the asymptotic expansion of $\phi_q^-(\xi)$ – if not in the lower half-plane $\mathrm{Im}\,\xi \le \omega'_+$, which turns out to be impossible – then in the strip $\mathrm{Im}\,\xi \in [\omega'_-, \omega'_+]$. The bounds, which we obtain in this way are mirror reflections of (5.16) and (5.17):

$$|\phi_q^+(\xi)| \le C, \tag{7.7}$$

$$|\partial_q^p\phi_q^+(\xi)| \le C|q|^{1-p}(|q|+|\xi|)^{-2}, \quad p \ge 1. \tag{7.8}$$

Notice that the bounds above are valid for q in the half-plane $\mathrm{Re}\, q \ge \sigma$ and ξ in the half-plane $\mathrm{Im}\,\xi \le \omega'_+$.

For $\phi_q^-(\xi)$, we use the Wiener-Hopf factorization formula:

$$\phi_q^-(\xi) = \frac{q}{q - i\mu\xi + \psi^0(\xi)}\frac{1}{\phi_q^+(\xi)},$$

which is valid in the strip $\operatorname{Im}\xi \in [\omega'_-, \omega'_+]$, and, therefore, derive estimates

$$|\phi_q^-(\xi)| \le C|q|/(|q|+|\xi|), \tag{7.9}$$

$$|\partial_q^p \phi_q^-(\xi)| \le C|q|/(|q|+|\xi|)^{p+1} \tag{7.10}$$

for ξ in this strip (and q in the half-plane $\operatorname{Re} q \ge \sigma$).

We will use $\widetilde{\phi_q^-}(\xi) = (1+i\xi)\phi_q^-(\xi)$ instead of $\phi_q^-(\xi)$; evidently,

$$|\widetilde{\phi_q^-}(\xi)| \le C(1+|\xi|)|q|/(|q|+|\xi|), \tag{7.11}$$

$$|\partial_q^p \widetilde{\phi_q^-}(\xi)| \le C(1+|\xi|)|q|^{1-p}/(|q|+|\xi|)^2 \tag{7.12}$$

7.3. **Simplification: the first step.** We simplify $\widetilde{V}(T,x)$ almost exactly as we simplified $V(T,x)$ in §5.2. We start with (5.21) with additional tildas atop of V's. The first, quite harmless, deviation is an additional factor $|q|$ in the bounds for $\tilde{G}$ and its derivatives because the bounds for $\phi_q^+(\xi)$ have this additional factor as compared to the case $\mu > 0$. The next one is more serious. Since the bounds for $\phi_q^-(\xi)$, hence, for $\phi_q^-(\xi)$, are available in a strip but not in a half-plane, we can prove that the terms $V_j(T,x)$, $j=2,3$, in (5.21) give contribution of class $H^{3/2-\epsilon}(\mathbb{R}_+)$ for any $\epsilon > 0$, but not of class $\mathring{H}^{3/2-\epsilon}(\mathbb{R}_+)$ for any $\epsilon > 0$. Therefore, at this stage, we cannot state that the error resulted from the omission of these terms is of order $O(x^{1-\epsilon})$, for any $\epsilon > 0$. Similar reservation must be made at the next stages of simplification. However, we will end with the formula with the two leading terms supported on $[0,+\infty)$, and the error terms of class $C^{1-\epsilon}(\mathbb{R}) \cup C^{2(1-\nu)}(\mathbb{R})$. Since the sum is supported on $[0,+\infty)$, the error term also does, and, therefore, the final conclusion will be the same.

Thus, at the end of Step 1 we obtain that it suffices to consider

$$\begin{aligned}\widetilde{V}_1(T,x) &= \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{qT}\partial_q^k\Big(q^{-1}\tilde{G}(q,0+)\\ &\quad\times\frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega'_+} e^{ix\xi}\widetilde{\phi_q^-}(\xi)(1+i\xi)^{-1}d\xi\Big)\,dq.\end{aligned} \tag{7.13}$$

Note that $\widetilde{\phi_q^-}(\xi)(1+i\xi)^{-1} = \phi_q^-(\xi)$.

7.4. **Step 2.** The same consideration, which we used to calculate the leading term of asymptotics, shows that, if k is sufficiently large, then, for any $s \in (0,\nu)$, we may introduce the factor $\mathbb{1}_{|q|\le|\xi|^s}$ under the integral sign, at the cost of an error of class $H^{3/2-\epsilon}(\mathbb{R})$. But on the set $|q| \le |\xi|^s$, we can use the standard improved formula for the Wiener-Hopf factor to find an efficient formula for the asymptotics of $\phi_q^-(\xi)$; in the vicinity of $q = i\mu\xi$, an efficient estimate does not exist. The standard formula is

$$\phi_q^-(\xi) = (1 - i\mu\xi/q)^{-1}\exp[\hat{I}^-(q,0) - \hat{I}^-(q,\xi)], \tag{7.14}$$

where $\hat{I}^-(q,\xi)$ is given by (5.3). Note that 1) to use (5.3), we will need to take $\omega_+ < \min\{1/(-\mu), \lambda_+\}$; 2) if $\tilde{\theta} > 0$ is sufficiently small, then we can transform the line of integration in (5.3) into the contour

$$\mathcal{L}_{\omega_+,\tilde{\theta}} = \{\eta = i\omega_+ + \rho e^{i\varphi} \mid \rho \ge 0, \varphi = \pi - \tilde{\theta} \text{ or } \varphi = \tilde{\theta}\}.$$

Next, we represent $\phi_q^-(\xi)$ in the form

$$\begin{aligned}\phi_q^-(\xi) \;=\; & (q/(-\mu))\exp[\hat{I}^-(q,0)](1+i\xi)^{-1} \\ & -(q/(-\mu))\exp[\hat{I}^-(q,0)](1+i\xi)^{-1}\hat{I}^-(1,\xi) \\ & -(q/(-\mu))\exp[\hat{I}^-(q,0)](1+i\xi)^{-1}[\hat{I}^-(q,\xi)-\hat{I}^-(1,\xi)] \\ & +(q/(-\mu))\exp[\hat{I}^-(q,0)](1+i\xi)^{-1}(1-\exp[\hat{I}^-(q,\xi)]+\hat{I}^-(q,\xi)) \\ & +(q/(-\mu))\exp[\hat{I}^-(q,0)-\hat{I}^-(q,\xi)][(-q/\mu+i\xi)^{-1}-(1+i\xi)^{-1}],\end{aligned}$$

and notice that, on the set (defined by) $|q| \le |\xi|^s$, all terms on the RHS but the first and second ones, and their derivatives of order p, admit estimates of the form $O(|q|^{1-p}(1+|\xi|)^{-1-\beta+\epsilon})$, where $\beta = \min\{2(1-\nu), 1\}$ and $\epsilon > 0$ is arbitrary.

As in the cases $\nu \in (0,1), \mu > 0$, and $\nu \in (1,2)$, which were considered in the preceding sections, the first two terms give rise to the leading term and second term of asymptotics, denote them $\widetilde{V}_{10}(T,x)$ and $\widetilde{V}_{11}(T,x)$. Substituting into (7.13), we find the leading term of the asymptotic formula (7.1):

$$\begin{aligned}\widetilde{V}_{10}(T,x) \;=\; & \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q^k \Big(q^{-1}\tilde{G}(q,0+) \\ & \times \frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega'_+} e^{ix\xi}(q/(-\mu))\exp[\hat{I}^-(q,0)](1+i\xi)^{-1}d\xi\Big)\, dq \\ =\; & \widetilde{\kappa}(T)e^{-x}\mathbb{1}_{(0,+\infty)}(x),\end{aligned} \tag{7.15}$$

where $\widetilde{\kappa}(T)$ is given by (7.2), and

$$\begin{aligned}\widetilde{V}_{11}(T,x) \;=\; & \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q^k \Big(q^{-1}\tilde{G}(q,0+) \\ & \times \frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega'_+} e^{ix\xi}(q/(-\mu))\exp[\hat{I}^-(q,0)](-\hat{I}^-(1,\xi))(1+i\xi)^{-1}d\xi\Big)\, dq.\end{aligned} \tag{7.16}$$

Define $F(x)$ by (5.26). The asymptotics of $F(x)$ has been calculated in the case $\mu > 0$ already. Since the sign of μ is not important in these calculations (although condition $\mu \ne 0$ is crucial), we can use (5.29) and (5.10). Substituting (5.29) into (7.16), we find

$$\widetilde{V}_{11}(T,x) = -\Gamma(\nu-1)\frac{d^0\sin(\gamma^0+\pi\nu/2)}{\pi\mu}\widetilde{\kappa}(T)x^{1-\nu} + O(x),$$

which is the second term in the asymptotic formula (7.1).

8. Numerical examples

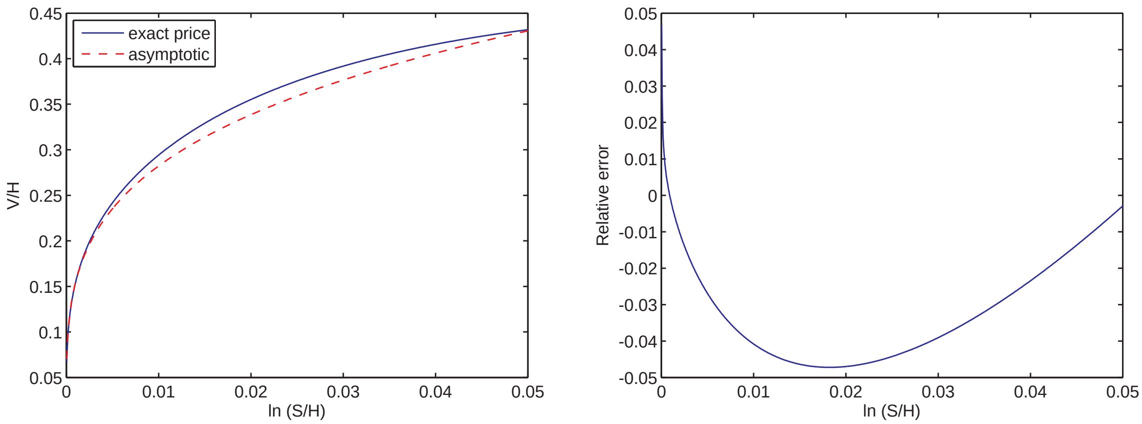
The following figures show numerical results for various types of processes and different choices of parameters. The "exact" prices were calculated using the method of [6, 7]. In all the examples listed the drift μ was determined by the EMM condition, which results in zero dividend rate. We consider the same down-and-out barrier put as in the Introduction, with barrier level $H = 2100$ and strike $K = 3500$. For each example, the rescaled price V/H is plotted, which means that the barrier is taken as a numeraire: $H = 1$, as throughout this paper. Some of the examples are taken from the literature, some are artificial, to illustrate the main qualitatively different cases. In each example, in the left panel, we plot the "exact price" calculated using Carr's randomization and the operator form of the Wiener-Hopf factorization method, the leading term of asymptotics, and, whenever available, the second term of asymptotics as well. In the right panel, we plot relative errors of the asymptotic prices w.r.t. the "exact one". It is natural to expect that the "exact prices" have most serious errors near the barrier, whereas the asymptotic prices are especially accurate near the barrier. The numerical examples indicate that the relative error of the leading term of asymptotics w.r.t. the "exact price" is very small near the barrier (less than 1%), which means that Carr's randomization together with the operator form of the Wiener-Hopf factorization method [6, 7] is very accurate indeed. The second term of asymptotics is necessary to reproduce (approximately) the shape of the price curve in the case, when the price is discontinuous at the barrier, and the leading term of asymptotics is a constant (the limit of the price as the underlying approaches the barrier). In other cases, the two-term asymptotic formula improves the performance of the one-term asymptotic formula but, typically, the improvement is not substantial. The reason is a rather complicated structure of the characteristic exponents of the majority of the standard Lévy models.

8.1. **NIG model.** For a NIG model, the agreement between the "exact price" and asymptotic price should be the best, and so it is. See Fig. 3a-3b, where we reproduce the same example as in the Introduction but with an additional panel for relative errors. The reader may notice that the relative errors are very small at a distance up to several percent from the barrier.

The quality of the asymptotic formula should be the best in this case, because the characteristic exponent of a NIG is especially well-behaved from the point of view of asymptotic analysis: as $\xi \to \infty$ in an appropriate strip of the complex plane,

$$\psi(\xi) = \psi_\infty|\xi| + O(1), \tag{8.1}$$

hence, the order of the O-term is less than the order of the leading term by 1; for other classes of Lévy processes of infinite activity, the difference of orders is smaller. But the asymptotic expansion of the symbol of the operator (in our case, $r - L = r + \psi(D)$), at infinity, translates into the asymptotic expansion of the solution of the boundary problem (here, the price), near the boundary (barrier). In more detail, in the case



(A) Price (B) Relative error

FIGURE 3. Example from [23] (same as in the Introduction), calibrated to market prices of vanilla options on Stoxx50E. NIG with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$, $\mu \approx 0.1607$, riskless rate $r = 0.03$, $T = 0.25$.

of NIG, both the jump part and the drift part are of the same order 1, and both are taken into account in the formula for the leading term of asymptotics of the characteristic exponent, hence, the Wiener-Hopf factors, hence, the price; for other classes of Lévy processes of infinite activity, either the jump part is the principal one, and the "drift term" is subordinate, or the other way round.

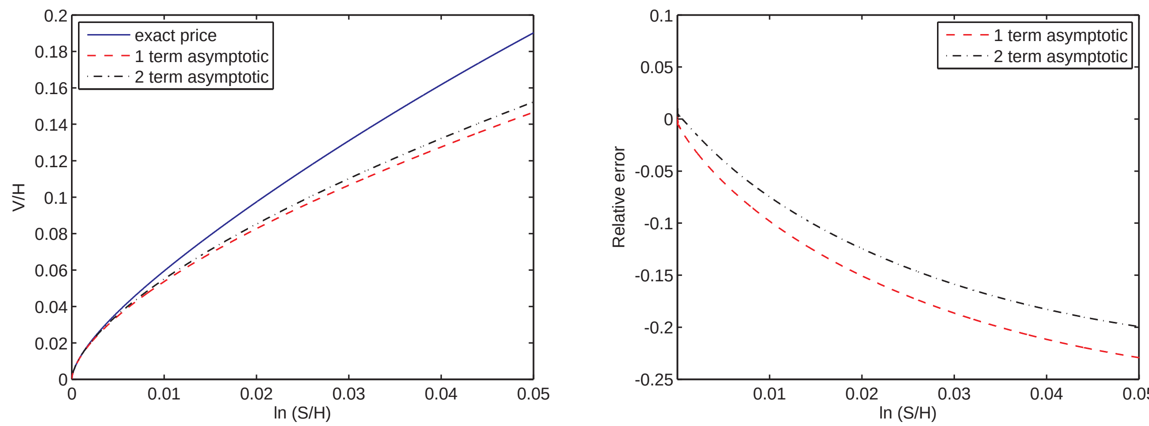
The next observation is that, very close to the barrier, the "exact price" is lower than the asymptotic one, which is natural because the effects of truncation and piecewise linear approximation in the pricing algorithm should lead to smaller prices, especially near the barrier. Hence, this example is in the perfect agreement with the theoretical expectations. Notice also, since the drift term is incorporated in the formula for the leading term of asymptotics, we cannot derive a simple two-term asymptotic formula for this case. However, this disadvantage is an advantage, in fact, because the two-term asymptotic formulas, which we derived, try to compensate for the failure of the leading term to take properly into account either the drift term (for processes of order $\nu > 1$) or the jump term (for processes of order $\nu \in (0, 1)$); but, in addition to this second term, one can derive several more terms, which can significantly influence the price. This effect can be inferred from the following figures.

8.2. **Processes of order $\nu \in (1, 2)$.** The next series of examples, for processes of order $\nu \in (1, 2)$, is more difficult to analyze because of the non-trivial interaction between the jump part (which, at the level of the infinitesimal generator, is the leading part) and the "drift part". For simplicity, we consider only the case $d^0_+ = d^0_- = d$, which includes NTS Lévy processes and symmetric KoBoL (a.k.a. CGMY) of order

$\nu \in (1, 2)$. In this case, the characteristic exponent has the following asymptotics

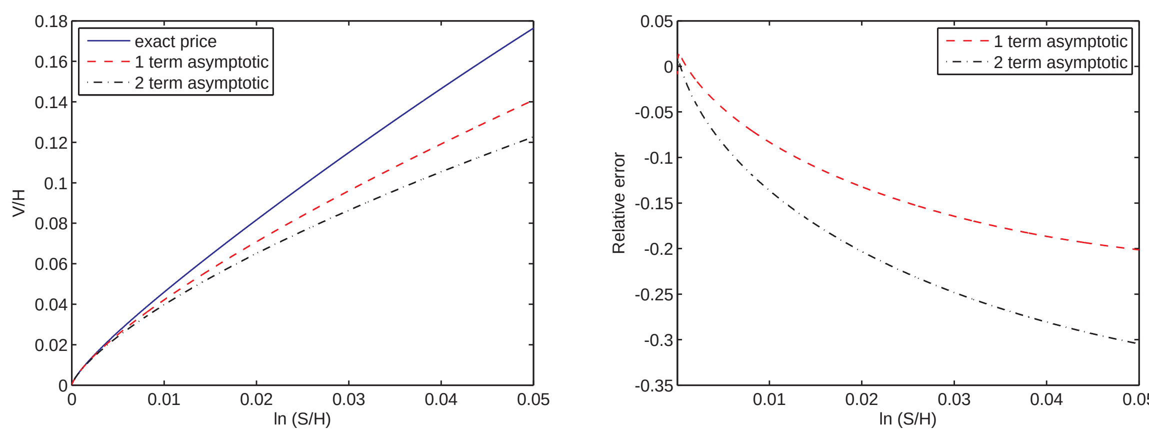
$$\psi(\xi) = d^0|\xi|^\nu - i\mu\xi + O(1), \tag{8.2}$$

as $\xi \to \infty$ in an appropriate strip around the real axis. The general observation



(A) Price (B) Relative error

FIGURE 4. KoBoL with $\nu = 1.25$, $\lambda_- = -9$, $\lambda_+ = 8$, $c = 0.25$, $\mu = 0.03$, riskless rate $r = 0.03$, $T = 0.25$.

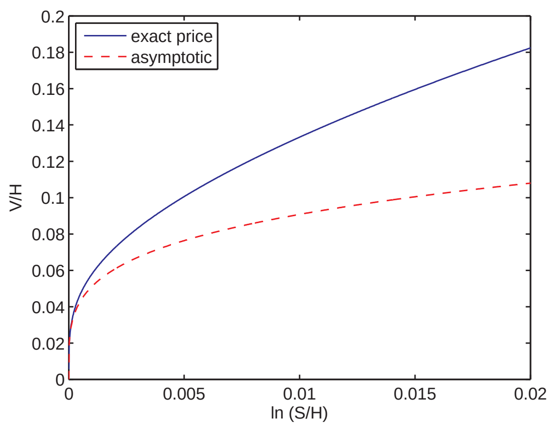


(A) Price (B) Relative error

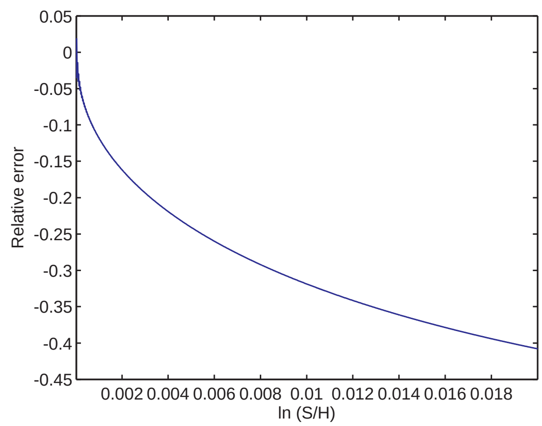
FIGURE 5. KoBoL with $\nu = 1.5$, $\lambda_- = -5$, $\lambda_+ = 10$, $c = 0.05$, $\mu \approx -0.1685$, riskless rate $r = 0.03$, $T = 0.25$.

is: the quality of the asymptotic formulas improves as the order ν increases, and the "drift" μ is getting smaller relatively to the parameter d. If $\nu - 1$ is small, then, in principle, one can derive additional terms of asymptotics, and improve the quality of the asymptotic formula. However, if the "drift parameter" μ is relatively large w.r.t. d^0, as in Fig. 5a-5b, then the difficulties are similar (and more serious) as in the theory of ordinary differential equations with a small parameter, where simple asymptotic formulas cannot work well, and more complicated asymptotic formulas must be derived[8].

8.3. **Processes of order** $\nu \in (0,1)$**.** For simplicity, we consider only the case $d^0_+ = d^0_- = d$, which includes NTS Lévy processes and symmetric KoBoL (a.k.a. CGMY) of order $\nu \in (0,1)$. If $\mu = 0$, then the quality of the asymptotic formula is similar to, but worse than the quality of the asymptotic formula for NIG because the order of the O-term in (8.2) is less than the order of the leading term by $\nu < 1$. See Fig. 6a-6b.



(A) Price

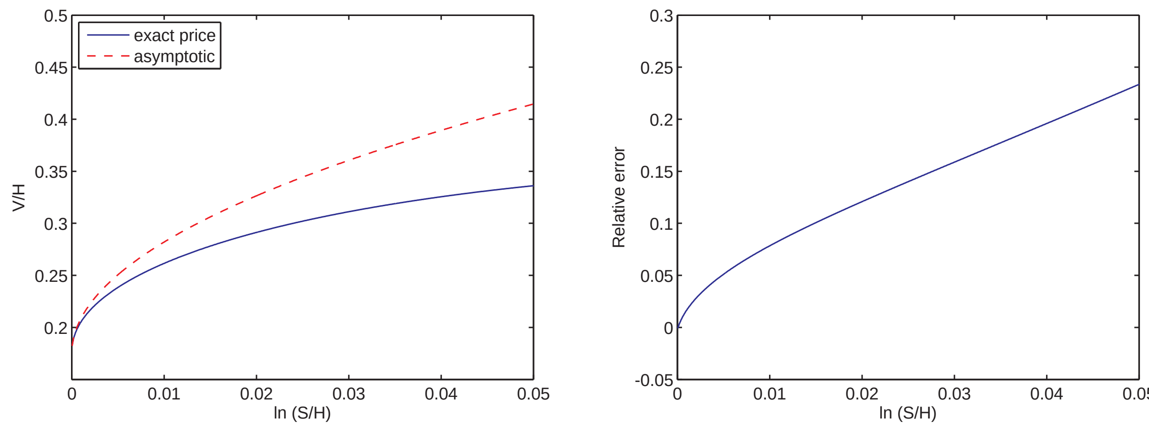


(B) Relative error

FIGURE 6. Example from [7]. KoBoL with $\nu = 0.5$, $\lambda_- = -8$, $\lambda_+ = 9$, $c = 1$, $\mu = 0$, riskless rate $r \approx 0.072309571491738$, $T = 0.25$.

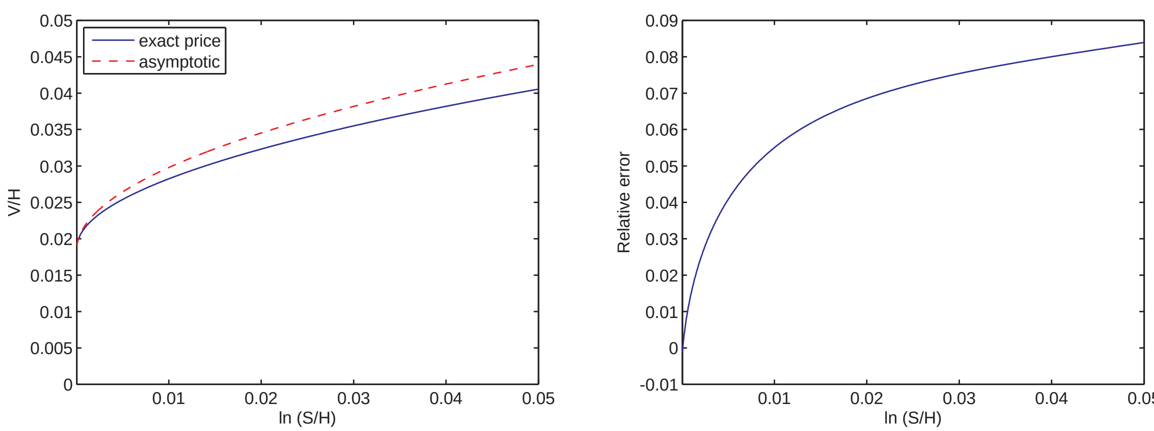
If $\mu \neq 0$, then we can derive two-term asymptotic formulas; naturally, the quality of the asymptotic formulas is good if $|\mu|/d^0$ is large. If $|\mu|$ is small relatively to d^0, then we are in an uncomfortable situation of the operator with a small parameter. We start with three examples taken from the literature; in all three cases, the drift is from the boundary, and the discontinuity of the price at the barrier is uncomfortably large. We do not plot the leading term of asymptotics, which is a constant; the reader can easily infer its value from the graph. See Fig. 7a-7b, 8a-8b, and 9a-9b.

[8]As the matter of fact, the proof of the asymptotic expansions in this paper contains these complicated formulas at one of the intermediate steps; but these formulas involve the inverse Fourier transform and are difficult to calculate.



(A) Price (B) Relative error

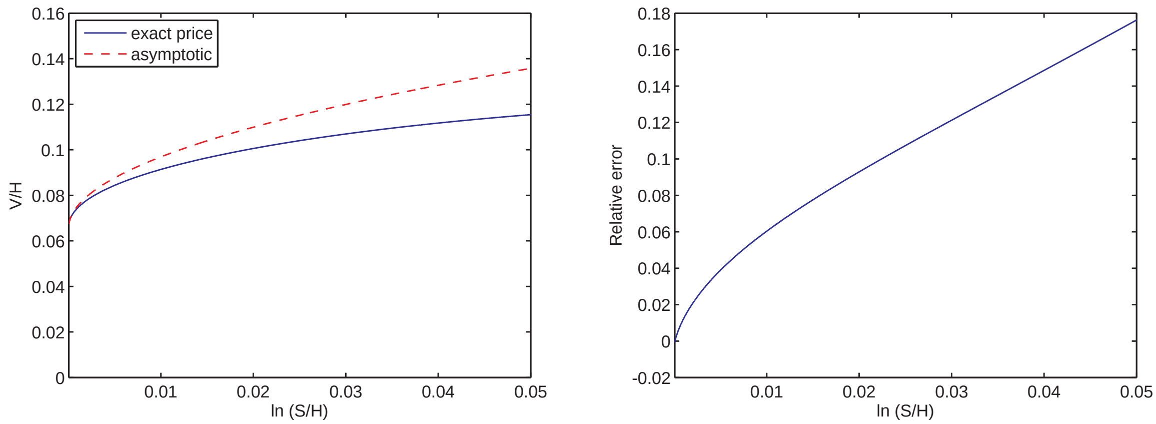
FIGURE 7. Example from [1]. KoBoL with $\nu = 0.5$, $\lambda_- = 10$, $\lambda_+ = 2$, $c = 0.5$, $\mu \approx 0.3257$, riskless rate $r = 0.05$, $T = 0.25$.



(A) Price (B) Relative error

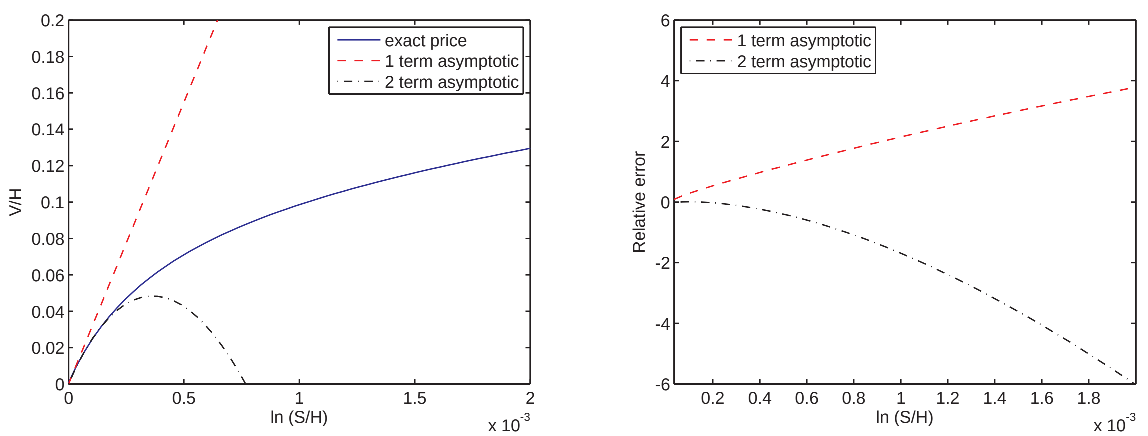
FIGURE 8. Example from [1], calibrated to vanilla options on Ford. KoBoL with $\nu = 0.5$, $\lambda_- = -11.0187$, $\lambda_+ = 1.9458$, $c = 0.6506$, $\mu \approx 0.4356$, riskless rate $r = 0.03$, $T = 1.5$.

In the case of the negative drift, the behavior of the price near the barrier is similar to the behavior of the price in the BM model. See Fig. 10a-10b and 11a-11b. Note that in Fig. 10a-10b, the drift is very small; as a result, the asymptotic formulas work well only in a tiny vicinity of the barrier.



(A) Price (B) Relative error

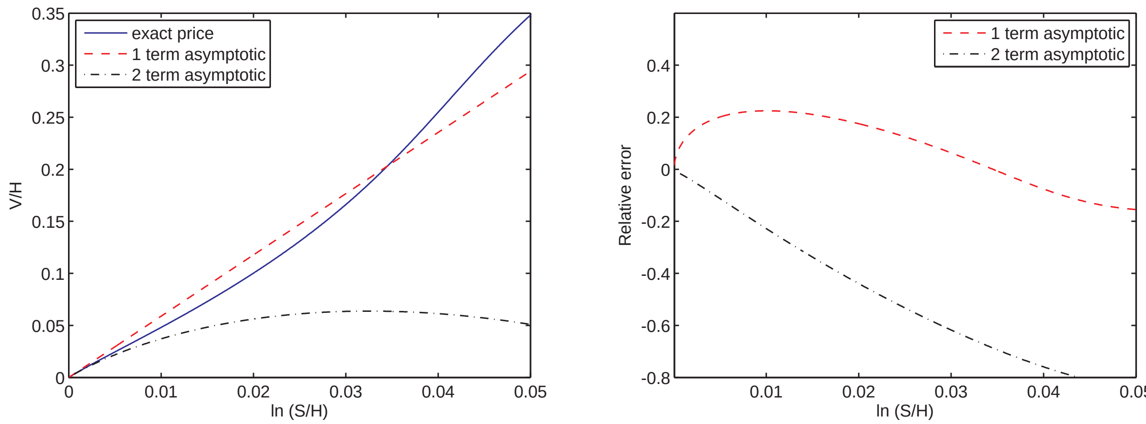
FIGURE 9. Example from [1], calibrated to vanilla options on General Motors. KoBoL with $\nu = 0.5$, $\lambda_- = -5.8031$, $\lambda_+ = 1.0084$, $c = 0.2171$, $\mu \approx 0.2006$, riskless rate $r = 0.03$, $T = 1.5$.



(A) Price (B) Relative error

FIGURE 10. Example from [20]. KoBoL with $\nu = 0.25$, $\lambda_- = -8$, $\lambda_+ = 9$, $c = 1$, $\mu \approx -0.0140$, riskless rate $r = 0.03$, $T = 0.25$.

8.4. **VG model.** In this case, we managed to calculate only the leading term of asymptotics in the case of the drift pointing from the boundary; this leading term is a positive constant. The reader may notice that the calibration to prices of vanilla options may lead to a rather large discontinuity of the price of a barrier option at the barrier. We will analyze this effect using empirical data in a future publication.



(A) Price (B) Relative error

FIGURE 11. KoBoL with $\nu = 0.5$, $\lambda_- = -2$, $\lambda_+ = 10$, $c = 0.25$, $\mu \approx -0.1803$, riskless rate $r = 0.05$, $T = 0.25$.

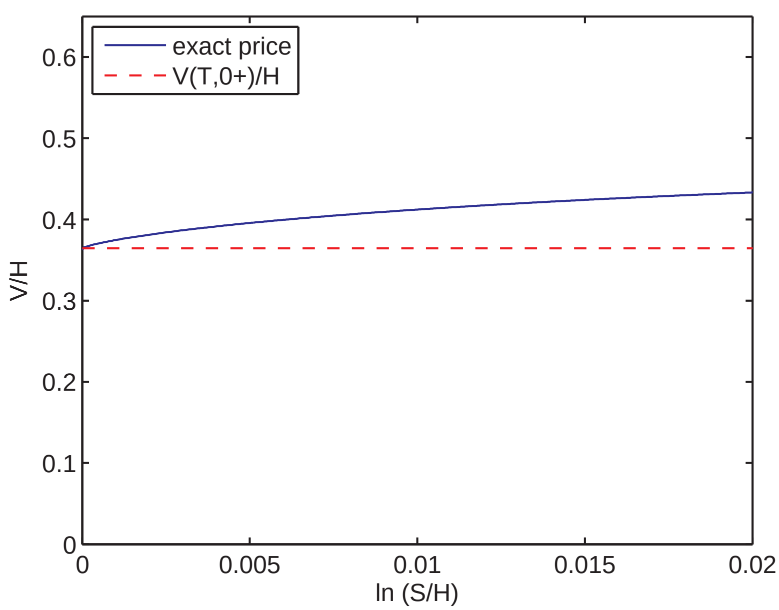


FIGURE 12. Example from [23], calibrated to market prices of vanilla options on Stoxx50E. VG with $\lambda_- = -11.876$, $\lambda_+ = 4.667$, $c = 0.925$, $\mu \approx 0.1282$, riskless rate $r = 0.03$, $T = 0.25$.

9. Conclusion

In the paper, we derived asymptotic formulas for the price of the down-and-out barrier option, of the form

$$V_{\mathrm{barr}}(T,S) = c(T)(S-H)^{\nu_-} + O((S-H)^{\nu_-+s}), \quad S\downarrow H, \tag{9.1}$$

where $\nu_- \in [0,1]$ and $s>0$ are determined by the parameters of the underlying process and the payoff function, and for the price of the first-touch digital option

$$V_{\mathrm{f.t.}}(T,S) = 1 - c(T)(S-H)^{\nu_-} + O((S-H)^{\nu_-+s}), \quad S\downarrow H. \tag{9.2}$$

Under additional conditions, the second term of asymptotics is derived as well. Ultimately, the validity of the asymptotics of the form (9.1) and (9.2) rests on the polynomial asymptotics of and the crucial representation (2.19) for the characteristic exponent at infinity, and the resulting asymptotic formulas for the Wiener-Hopf factors (1.1). The representation is valid and functions that appear in the integral representations (1.3) and (1.6) are well-defined for wide classes of regular Lévy processes of exponential type introduced in [11] and the VG model (with a certain reservation about the latter: see below); however, the proof simplifies if we consider a subclass of strongly regular Lévy processes of exponential type (sRLPE) introduced in §2.3, and derive different representations for the Wiener-Hopf factors, which may be of an independent interest.

Class of sRLPEs contains BM, Kou's model and HEJD model (processes of order $\nu=2$); processes of KoBoL family (a.k.a. CGMY model) given by (2.5), of order $\nu\in(0,2), \nu\neq 1$; NTS processes of order $\nu\in(0,2)$ given by (2.8), a special case being NIG (NTS process of order $\nu=1$); and VG model: sRLPE of order $\nu=0+$. In §2.5, the reader can find explicit formulas for the exponent $\nu_\pm$, which define the order of asymptotics. The results for up-and-out options can be obtained by the mirror symmetry, with ν_+ playing the part of ν_-.

The main qualitatively different cases are

(1) $\nu=2$ (a process with the non-trivial BM component), or $\nu\in(0,1)$ and $\mu<0$ (processes of finite variation and infinite activity, with the drift pointing to the boundary); then $\nu_-=1$, and the graph of the price near the boundary looks similar to the graph in the BM case;
(2) KoBoL processes of order $\nu\in(1,2)$ in the symmetric case $c_+=c_-$ in (2.5) and NTS processes of order $\nu\in(1,2)$; then $\nu_\pm=\nu/2\in(0,1)$;
(3) KoBoL processes of order $\nu\in(1,2)$ in the asymmetric case $c_+\neq c_-$; then any $\nu_\pm\in(0,\nu)$ are possible (such that $\nu_++\nu_-=\nu$);
(4) NIG: depending on the "drift" μ, any $\nu_\pm\in(0,1)$ are possible (s.t. $\nu_++\nu_-=1$);
(5) driftless KoBoL processes of order $\nu\in(0,1)$ in the symmetric case and NTS processes of order $\nu\in(0,1)$; then $\nu_\pm=\nu/2\in(0,1/2)$;
(6) driftless moderately asymmetric KoBoL processes of order $\nu\in(0,1)$; depending on c_+/c_-, any $\nu_\pm\in(0,\nu)$ are possible (such that $\nu_++\nu_-=\nu$);

(7) processes of order $\nu \in [0+, 1)$ with the drift pointing from the boundary: $\mu > 0$; then $\nu_- = 0$ and $\nu_+ = 1$ so that the prices of the down-and-out barrier options and first-touch digitals are discontinuous at the barrier (but the leading term of the asymptotics of the price of the up-and-out option is of the same form as in the BM case)

The reader may have noticed that the list above does not include the VG model with $\mu \le 0$. In this case, we are able to prove only that the price is continuous at the boundary. We conjecture that in the case $\mu = 0$, the price decays slower than any power of $S - H$, but the situation is less clear in the case $\mu < 0$, so we prefer to avoid making a guess.

Finally, we found that,

- in some empirical studies, fitting using vanillas leads to parameter values such that there must be a large positive limit of the price of the barrier option at the boundary;
- the numerical method based on Carr's randomization and the operator form of the Wiener-Hopf factorization [6, 7] is in a good agreement with the asymptotic price in a tiny vicinity of the barrier, which implies that the method is very accurate.

Appendix A. Elements of the theory of generalized functions [21, 11]

The Sobolev space $H^s(\mathbb{R})$ consists of generalized functions with the finite norm

$$||u||_s = \left(\int_{\mathbb{R}} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2},$$

where $\hat{u}$ is the Fourier transform of u. $\mathring{H}^s(\mathbb{R}_+)$ denotes the subspace of functions from $H^s(\mathbb{R}_+)$ supported on $[0, +\infty)$. For any $s \in (-1/2, 1/2)$, the multiplication-by-$\mathbb{1}_{(0,+\infty)}$ operator extends to a bounded operator $\mathbb{1}_{(0,+\infty)} : H^s(\mathbb{R}) \to \mathring{H}^s(\mathbb{R}_+)$.

If a is a measurable function, which admits an estimate

$$|a(\xi)| \le C(1 + |\xi|)^m, \quad \xi \in \mathbb{R}, \tag{A.1}$$

then the PDO $a(D) : H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ is a bounded operator, with the norm bounded by C, and if (A.1) holds for ξ in the half-plane $\operatorname{Im} \xi \le 0$, then $a(D)$ maps $\mathring{H}^s(\mathbb{R}_+)$ to $\mathring{H}^{s-m}(\mathbb{R}_+)$.

The Sobolev embedding theorem (in 1D) states that if $s > 1/2$, then, for any $\epsilon > 0$, $H^s(\mathbb{R})$ is continuously embedded in the Hölder space $C^{s-1/2-\epsilon}(\mathbb{R})$. Denoting by $\mathring{C}^s(\mathbb{R}_+)$ the subspace of $C^s(\mathbb{R})$ of functions vanishing on $(-\infty, 0]$, we have $\mathring{H}^s(\mathbb{R}) \subset \mathring{C}^{s-1/2-\epsilon}(\mathbb{R}_+)$, for any $s > 1/2$.

If $u \in H^s(\mathbb{R})$, and $s > m - 1/2$, where m is a positive integer, then

$$\begin{aligned}\mathbb{1}_{(0,+\infty)}u &= u(0+)(1+iD)^{-1}\delta + ((1+iD)u)(+0)(1+iD)^{-2}\delta \\ &\quad \cdots + (1+iD)^{-m}\mathbb{1}_{(0,+\infty)}(1+iD)^m u \\ &= u(0+)(1+iD)^{-1}\delta + (u(0+)+u'(+0))(1+iD)^{-2}\delta \\ &\quad \cdots + (1+iD)^{-m}\mathbb{1}_{(0,+\infty)}(1+iD)^m u \qquad \text{(A.2)}\end{aligned}$$

(see [21, Lemma 5.5],[11, Theorem 15.15]).

Appendix B. Digital puts and calls, and the case of strike $K < H$

If strike $K < H$, then only the down-and-out call option makes sense, and we may assume that $G(x) = e^x - K$ and calculate $\tilde{G}(q,x) = \phi_q^+(-i)e^x - K$ explicitly. The rest of the calculations remain the same. Similarly, in the case of the call option with strike $K < H$, $\tilde{G}(q,x) = 1$. In the case of digitals with strike $K > H(=1)$, (3.1) fails. We take $\chi \in C^\infty(\mathbb{R})$, $\chi(x) = 1, x < \ln K/2, \chi(x) = 0, x > \ln K/3$, and write (A.2) in the form

$$\begin{aligned}\mathbb{1}_{(0,+\infty)}\tilde{G}(q,x) &= \tilde{G}(q,0+)(1+iD)^{-1}\delta + \cdots + ((1+iD)^m\tilde{G})(q,0+)(1+iD)^{-m}\delta \\ &\quad +(1+iD)^{-m}\mathbb{1}_{(0,+\infty)}(1+iD)^m\chi(x)\tilde{G}(q,x) + (1-\chi(x))\tilde{G}(q,x). \qquad \text{(B.1)}\end{aligned}$$

The last term vanishes in a neighborhood of zero, hence, it is irrelevant for the study of the leading term of asymptotics. For the standard classes of RLPEs, the characteristic exponent admits estimates of the form

$$|\psi^{(s)}(\xi)| \le C_s(1+|\xi|)^{\nu-s}, \quad s = 0, 1, \dots,$$

and the Wiener-Hopf factors admit similar estimates. Using the estimates for $\phi_q^+(\xi)$ and integrating by parts in the oscillatory integral that defines $\chi(x)\tilde{G}(q,x)$ (composition of the Fourier transform, multiplication-by-$\phi_q^+(\xi)$ and the inverse Fourier transform applied to $G(x)$), one easily obtains that $\chi(x)\tilde{G}(q,x)$ is of class $C^\infty(\mathbb{R})$ in x, with the derivatives w.r.t. q decaying faster with each differentiation.

Hence, in (B.1), all terms but the first one do not influence the leading term of asymptotics. These terms do not influence the second term as well, if the second term is of order x^{ν_-+s}, where $s \in (0,1)$. If the second term is of order x^{ν_-+1}, then the second term in (B.1) must be taken into account (in the present paper, we do not consider this complicated situation).

Appendix C. Wiener-Hopf factorization

C.1. **Three forms of the Wiener-Hopf factorization.** The proof of (3.7) in [7] is very close to the proof of the Wiener-Hopf factorization formula in the form used in probability (see, e.g., [35, p. 98]):

$$\mathbb{E}\big[e^{X_{T_q}}\big] = \mathbb{E}\big[e^{\overline{X}_{T_q}}\big] \cdot \mathbb{E}\big[e^{\underline{X}_{T_q}}\big]. \qquad \text{(C.1)}$$

The operator form of the Wiener-Hopf factorization

$$\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^- \tag{C.2}$$

can also be proved similarly to (C.1). Finally, introducing the notation

$$\phi_q^+(\xi) = \mathbb{E}\big[e^{i\xi \overline{X}_{T_q}}\big], \qquad \phi_q^-(\xi) = \mathbb{E}\big[e^{i\xi \underline{X}_{T_q}}\big]. \tag{C.3}$$

and noticing that

$$\mathbb{E}\big(e^{iX_{T_q}\xi}\big) = \frac{q}{q+\psi(\xi)}, \tag{C.4}$$

we can write (C.1) in the form

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi). \tag{C.5}$$

Equation (C.5) is a special case of the factorization of functions on the real line into a product of two functions analytic in the upper and lower open half planes and admitting the continuous continuation up to the real line. This is the initial factorization formula discovered by Wiener and Hopf [38] in 1931, for functions of a much more general form than in the LHS of (C.5).

C.2. **Realization of the EPV operators using the Fourier transform.** Decomposing a sufficiently regular function $f(x)$ as a Fourier integral and using (C.4), we obtain

$$(\mathcal{E}_q f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} \frac{q\widehat{f}(\xi)}{q+\psi(\xi)}\, d\xi, \tag{C.6}$$

where $\widehat{f}(\xi) = \mathcal{F}_{x\to\xi} f$ is the Fourier transform of f. Identity (C.6) can be justified under rather weak regularity assumptions; we refer the reader to [11, §2.3.3] for the details (with the notation of *loc. cit.*, we have $\mathcal{E}_q = qU^q$, where U^q is referred to as the resolvent operator, or the q-potential operator, of X).

Similarly, it follows from (3.5)–(3.6) and (C.3) that

$$(\mathcal{E}_q^\pm f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\phi_q^\pm(\xi)\widehat{f}(\xi)\, d\xi. \tag{C.7}$$

We conclude that $\mathcal{E}_q$ and $\mathcal{E}_q^\pm$ are pseudo-differential operators (PDOs) with the symbols $q/(q+\psi(\xi))$ and $\phi_q^\pm(\xi)$, respectively.

To realize PDOs $\mathcal{E}_q^\pm$, one needs explicit analytical expressions for the Wiener-Hopf factors $\phi^\pm(\xi)$. There exist general formulas [36, Eqns. (4.5), (4.6)] for any Lévy process but they are inefficient both for theoretical study and computational purposes.

Under a certain regularity assumption on the characteristic exponent $\psi(\xi)$ of X (see, e.g., [11, Theorem 3.2]), S.I. Boyarchenko and Levendorskiĭ obtained integral formulas for the Wiener-Hopf factors $\phi^\pm_q(\xi)$, which are efficient for computational purposes. This assumption holds in all model examples of Lévy processes of exponential type, including those listed in §2.4, so we prefer not to state it to save space. To state these formulas, recall that (see [11, Eqn. (3.40)]) if the characteristic exponent is analytic in a strip (λ_-, λ_+) and increases not faster than a polynomial in this strip, then, for any $q > 0$, there exist $\omega_- < 0 < \omega_+$ and $\delta > 0$ such that

$$\mathrm{Re}(q + \psi(\xi)) \geq \delta, \quad \mathrm{Im}\,\xi \in [\omega_-, \omega_+]. \tag{C.8}$$

The formulas [11, Eqns. (3.58), (3.60)] are as follows: for $\pm\,\mathrm{Im}\,\xi > \pm\omega_\mp$,

$$\phi^\pm_q(\xi) = \exp\left[\pm\frac{1}{2\pi i}\int_{\mathrm{Im}\,\eta=\omega_\mp} \frac{\xi \cdot \ln(q + \psi(\eta))}{\eta(\xi - \eta)}\,d\eta\right]. \tag{C.9}$$

Note that one can take any $\omega_- \in (\lambda_-, 0)$ and $\omega_+ \in (0, \lambda_+)$, and, after that, choose a sufficiently large $q > 0$ so that (C.8) holds. Next, to apply the inverse Laplace transform w.r.t q, as in the main body of the paper, we need to consider the analytic continuation w.r.t. q as well. In addition, we need to obtain estimates for $\phi^\pm_q(\xi)$, and, whenever possible, calculate the leading term of asymptotics of $\phi^\pm_q(\xi)$ as $\xi \to \infty$. It follows from (C.8) that if $\sigma > 0$ is sufficiently large then the RHS in (C.9) defines the analytic continuation into the half-plane $\mathrm{Im}\,q \geq \sigma$. However, it is advantageous to extend $\phi^\pm_q(\xi)$ w.r.t. q into an obtuse sector $\Sigma_{\sigma,\theta} = \{q = \sigma + \rho e^{i\varphi} | -\theta \leq \varphi \leq \theta,\ \rho \geq 0\}$, where $\sigma > 0$, $\theta \in (\pi/2, \pi)$, the reason being that we can transform the integral in the Laplace inversion formula into the integral over the contour $\partial\Sigma_{\sigma,\theta} = \{q = \sigma + \rho e^{i\varphi} | \phi = \pm\theta,\ \rho \geq 0\}$, and e^{Tq} will exponentially decay as $q \to \infty$, $q \in \partial\Sigma_{\sigma,\theta}$. This is very useful because the rate of convergence of the integral improves significantly.

Since formulas (C.9) are insufficient to derive the leading term of asymptotics of the price near the barrier, we need to derive different realizations of these formulas. The improved formulas for the factors depending on the order of the process, and, for processes of order $\nu \leq 1$, on the drift as well, we have to consider several cases.

In the main body of the paper, we fix $\lambda_- < \omega_- < \omega'_- < 0 < \omega'_+ < \omega_+ < \lambda_+$, and derive the formulas and estimates for $\pm\,\mathrm{Im}\,\xi \geq \pm\omega'_\mp$; the estimates are uniform for these ξ and $\mathrm{Re}\,q \geq \sigma$, where $\sigma > 0$ is large and fixed. In some cases, which are indicated explicitly, the results are valid for $q \in \Sigma_{\sigma,\theta}$ for some $\theta \in (\pi/2, \pi)$.

Appendix D. Leading term of asymptotics: Cases $\nu \in (0, 1), \mu = 0$

We start with a necessary modification of the explicit formulas for the Wiener-Hopf factors, then use this modification to establish estimates for the factors and their derivatives, and, finally, explain how these estimates allow us to justify the same main steps in the proof of the asymptotic formula.

D.1. **The Wiener-Hopf factorization: Case $\nu \in (0,1), \mu = 0$.** We consider the case when $\gamma := |\gamma_\pm| = |\arg d_\pm|$ is small so that $\gamma + \pi\nu/2 < \pi/2$ and $\nu_\pm = \nu/2 \mp \gamma/\pi > 0$. These conditions are satisfied for NTS ($d_\pm = \delta$, hence, $\gamma = 0$) and for KoBoL if c_+/c_- is sufficiently close to 1 (Indeed, if $c_+ = c_-$, then $\gamma = 0$, and $\gamma + \pi\nu/2 < \pi/2$ since $\nu < 1$; since γ depends continuously on c_+/c_-, the inequality $\gamma + \pi\nu/2 < \pi/2$ holds if $c_+/c_- - 1$ is sufficiently small. Similarly, $\nu_\pm > 0$ if $\gamma < \pi\nu/2$).

If $\gamma + \pi\nu/2 < \pi/2$, then there exists $\delta > 0$ such that for sufficiently large σ, and all ξ in the strip $\mathrm{Im}\,\xi \in [\omega'_-, \omega'_+]$, we have $\sigma + \psi(\xi) \in \Sigma_{0,\pi/2-\delta} \setminus 0$. Take $\theta \in (\pi/2, \pi/2+\delta)$. Then $((\Sigma_{0,\pi/2-\delta} \setminus 0) + \Sigma_{0,\theta}) \cap (-\infty, 0] = \emptyset$. Hence, for all $q \in \Sigma_{\sigma,\theta}$, $\rho \geq 0$ and $\varphi \in [0, \pi/2 - 0] \cup [\pi/2 + 0, \pi] \cup [-\pi, -\pi/2 - 0] \cup [-\pi/2 + 0, 0]$,

$$\arg(q + \psi(\rho e^{i\varphi})) \in (-\pi, \pi). \tag{D.1}$$

However, even for q on the line $\mathrm{Re}\, q = \sigma$, $|\arg q^{1/\nu}| > \pi/2$ if $|q|$ is large enough. Hence, we cannot use the improved factorization. So, we use the initial form (C.9), and, by a slight abuse of notation, we denote the integral in (C.9) by $I^\pm(q,\xi)$. On the strength of (D.1), we can transform the integral in (C.9) to the cut $i(-\infty, \lambda_-]$ (in the case "+") and to the cut $i[\lambda_+, +\infty)$ (in the case "–"). The results are

$$I^+(q,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\lambda_-} \frac{\xi \ln[(q + \psi(iz-0))/(q+\psi(iz+0))]dz}{z(z+i\xi)} \tag{D.2}$$

$$I^-(q,\xi) = \frac{1}{2\pi}\int_{\lambda_+}^{+\infty} \frac{\xi \ln[(q + \psi(iz-0))/(q+\psi(iz+0))]dz}{z(z+i\xi)} \tag{D.3}$$

Below, we derive bounds for $I^\pm(q,\xi)$ and $\phi_q^\pm(\xi)$, for q in the half-plane $\mathrm{Re}\, q \geq \sigma$ and ξ in the half-plane $\pm \mathrm{Im}\,\xi \geq \pm\omega'_\mp$, and calculate the asymptotics of $\phi_q^\pm(\xi)$ as $\xi \to \infty$ and $|q| \leq |\xi|^s$, where $s \in (0,\nu)$. The arguments for "+" and "–" signs being similar, we give the proofs for sign "+".

D.2. **Boundedness.** We have $\psi(iz \mp 0) = d_\mp e^{\pm i\pi\nu/2}(-z)^\nu + O(1)$, as $z \to -\infty$, therefore, for $|q| \ll (-z)^\nu$,

$$\frac{q + \psi(iz-0)}{q+\psi(iz+0)} = \frac{d_-}{d_+}e^{i\pi\nu} + O(|q|/|z|^\nu). \tag{D.4}$$

For $(-z)^\nu \ll |q|$,

$$\frac{q + \psi(iz-0)}{q+\psi(iz+0)} = 1 + O(|z|^\nu/|q|). \tag{D.5}$$

Recall that $2\nu_+ = \nu - 2\gamma_+/\pi = \nu + (\gamma_- - \gamma_+)/\pi$. Hence, for $z \leq \lambda_-$, we can rewrite (D.4) as

$$\Phi^+(q,z) := \frac{q + \psi(iz-0)}{q+\psi(iz+0)}e^{-i2\pi\nu_+} = 1 + O(|q|/|z|^\nu). \tag{D.6}$$

Now, we divide $(-\infty, \lambda_-]$ into two parts $(-\infty, -|q|^{1/\nu})$ and $[-|q|^{1/\nu}, \lambda_-]$, and denote the corresponding integrals $I_j^+(q,\xi)$, $j = 1, 2$. Then, to take advantage of (D.5), we represent $I_1^+(q,\xi)$ as the sum

$$I_1^+(q,\xi) = I_{11}^+(q,\xi) + I_{12}^+(q,\xi),$$

where

$$I_{11}^+(q,\xi) := \frac{1}{2\pi}\int_{-\infty}^{-|q|^{1/\nu}} \frac{\xi 2\pi\nu_+ i\, dz}{z(z+i\xi)} = \ln\left[(1 - i\xi/|q|^{1/\nu})^{-\nu_+}\right], \tag{D.7}$$

$$I_{12}^+(q,\xi) = \frac{1}{2\pi}\int_{-\infty}^{-|q|^{1/\nu}} \frac{\xi \ln \Phi^+(q,z)dz}{z(z+i\xi)}. \tag{D.8}$$

Below, we show that $I_2^+(q,\xi)$ and $I_{12}^+(q,\xi)$ are uniformly bounded; hence, from (D.7), the following estimate holds for sign "+"

$$c(1+|\xi|/|q|^{1/\nu})^{-\nu_\pm} \le |\phi_q^\pm(\xi)| \le C(1+|\xi|/|q|^{1/\nu})^{-\nu_\pm}. \tag{D.9}$$

We prove the (uniform) boundedness of $I_2(q,\xi)$ first. The argument similar to the proof of the first line in (F.4) shows that if $|z| \le |q|^{1/\nu}$, then

$$\ln \frac{q+\psi(iz-0)}{q+\psi(iz+0)} = O(|z|^\nu/|q|).$$

Hence,

$$|I_2^+(q,\xi)| \le C\int_{-|q|^{1/\nu}}^{\lambda_-} \frac{|\xi||z|^\nu|q|^{-1}}{|z|(|z|+|\xi|)}\,dz \le C|q|^{-1}|z|^\nu \Big|_{-|q|^{1/\nu}}^{\lambda_-} \le C.$$

Now we consider $I_{12}^+(q,\xi)$. An argument similar to the proof of the second line in (F.4) shows that if $|z|^\nu \ge |q|$, then the logarithm of the LHS in (D.6) is bounded (in absolute value) by $C|q||z|^{-\nu}$, hence

$$|I_{12}^+(q,\xi)| \le C\int_{-\infty}^{-|q|^{1/\nu}} \frac{|\xi||q||z|^{-\nu}}{|z|(|z|+|\xi|)}dz.$$

If $|\xi| \le |q|^{1/\nu}$, then we estimate further via

$$C|\xi||q|\int_{-\infty}^{-|q|^{1/\nu}} |z|^{-\nu-2}dz \le C_1|\xi||q|^{-1/\nu} \le C_1;$$

if $|\xi| \ge |q|^{1/\nu}$, we have

$$\begin{aligned} |I_{12}^+(q,\xi)| &\le C_1\left[\int_{-|\xi|}^{-|q|^{1/\nu}} |q||z|^{-\nu-1}dz + \int_{-\infty}^{-|\xi|} |\xi||q||z|^{-\nu-2}dz\right] \\ &\le C_2\left[1 + |\xi||q||\xi|^{-\nu-1}\right] \le C_3. \end{aligned}$$

The uniform boundedness of $I_2^+(q,\xi)$ and $I_{12}^+(q,\xi)$ hence, (D.9) have been proved, and (D.9) follows (for sign "+"; the proof for sign "–" is by symmetry).

Notice that in the previous constructions

$$\arg(\sigma+\psi(iz\mp 0))\in\left[-\frac{\pi}{2}+\delta,\frac{\pi}{2}-\delta\right],$$

where $\delta>0$. Hence, if we allow for $q\in\Sigma_{\sigma,\theta}$, where $\theta\in(\pi/2,\pi/2+\delta)$, then all the arguments above remain valid. However, unlike the case $\nu>1$, we do not have a factorization of $\phi^\pm_q(\xi)$ into a product of a bounded factor and a factor, which controls the rate of decay at infinity, such that both factors are analytic.

D.3. **Estimates for the derivatives.** As above, we consider the plus sign. Differentiating w.r.t. q under the integral sign in (D.2), we obtain, for $k\ge 1$,

$$\partial^k_q I^+(q,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\lambda_-}\frac{\xi\,\partial^{k-1}_q D\Psi(q,z)}{z(z+i\xi)}dz, \tag{D.10}$$

where

$$D\Psi(q,z):=\partial_q\ln\frac{q+\psi(iz-0)}{q+\psi(iz+0)}=\frac{\psi(iz+0)-\psi(iz-0)}{(q+\psi(iz-0))(q+\psi(iz+0))}$$

admits an estimate

$$\partial^{k-1}_q D\Psi(q,z)=O(|z|^\nu(|q|+|z|^\nu)^{-k-1}). \tag{D.11}$$

Hence,

$$|\partial^k_q I^+(q,\xi)| \ \le\ C_1\int_{-\infty}^{\lambda_-}\frac{|\xi||z|^{\nu-1}dz}{(|q|+|z|^\nu)^{k+1}(|z|+|\xi|)}\le C_2|q|^{-k}\int_{-\infty}^{0}\frac{|z|^{\nu-1}}{(1+|z|^\nu)^{k+1}}dz,$$

where C_1,C_2 depend on k. A similar estimate holds for sign "–", and, since $\nu<1$, we obtain

$$|\partial^k_q I^\pm(q,\xi)|\le C_k|q|^{-k}, \tag{D.12}$$

and, using (D.9), conclude that

$$|\partial^k_q\phi^\pm_q(\xi)|\le C_k|q|^{-k}(1+|\xi|/|q|^{1/\nu})^{-\nu_\pm}. \tag{D.13}$$

D.4. **Estimates on the subset** $|q|\le|\xi|^s$**,** $s\in(0,\nu)$**.** Consider sign "–". We modify the constructions in the previous subsection inserting the factor $e^{-2\pi\nu_+ i}$ into (D.2), that is, replacing the fraction under the logarithm sign with $\Phi^+(q,z)$, and introducing

$$I^{++}(q,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\lambda_-}\frac{\xi\ln\Phi^+(q,z)dz}{z(z+i\xi)}. \tag{D.14}$$

A calculation similar to (D.7) gives

$$\phi^+_q(\xi)=(1+i\xi/\lambda_-)^{-\nu_+}\exp I^{++}(q,\xi). \tag{D.15}$$

Next, we represent $I^{++}(q,\xi)$ as the difference

$$I^{++}(q,\xi)=\hat I^{++}(q,0)-\hat I^{++}(q,\xi), \tag{D.16}$$

where

$$\hat{I}^{++}(q,\xi)=\frac{1}{2\pi i}\int_{-\infty}^{\lambda_-}\frac{\ln\Phi^+(q,z)dz}{z+i\xi}. \tag{D.17}$$

On the set (defined by) $|z|^\nu\le|q|$, $\ln\Phi^+(q,z)$ is bounded, and on the set $|z|^\nu\ge|q|$, $\ln\Phi^+(q,z)=O(|q||z|^{-\nu})$ (see (D.6)). Hence,

$$\begin{aligned}\left|\hat{I}^{++}(q,\xi)\right| &\le C_1\left(\int_{-|q|^{1/\nu}}^{\lambda_-}\frac{dz}{|z|+|\xi|}+\int_{-|\xi|}^{-|q|^{1/\nu}}\frac{|q||z|^{-\nu}dz}{|z|+|\xi|}+\int_{-\infty}^{-|\xi|}\frac{|q||z|^{-\nu}dz}{|z|+|\xi|}\right)\\ &\le C_2\left(|\xi|^{-1}|q|^{1/\nu}+|\xi|^{-1+1-\nu}|q|+|q||\xi|^{-\nu}\right).\end{aligned}$$

Since $\nu<1$, the last sum is bounded by $C|\xi|^{s-\nu}$. To obtain the estimates for the derivatives, we use (D.11):

$$\begin{aligned}\left|\partial_q^k\hat{I}^{++}(q,\xi)\right| &\le C_1\left(\int_{-|q|^{1/\nu}}^{\lambda_-}+\int_{-\infty}^{-|q|^{1/\nu}}\right)\frac{|z|^\nu dz}{(|q|+|z|^\nu)^{k+1}(|z|+|\xi|)}\\ &\le C_2\left(|q|^{-k-1}|\xi|^{-1}|z|^{\nu+1}|_{z=-|q|^{1/\nu}}+|\xi|^{-1}|z|^{-k\nu+1}|_{z=-|q|^{1/\nu}}\right)\\ &\le C_3|q|^{-k}|q|^{1/\nu}|\xi|^{-1}\\ &\le C_3|q|^{-k}|\xi|^{(s-\nu)/\nu}.\end{aligned}$$

Since $\nu<1$, we conclude that for $k\in\mathbb{Z}_+$,

$$|\partial_q^k\hat{I}^{++}(q,\xi)|\le C|q|^{-k}|\xi|^{s-\nu}. \tag{D.18}$$

It follows from (D.15), (D.16) and (D.18) that, on the set $|q|\le|\xi|^s$,

$$\phi_q^+(\xi)=(-\lambda_-)^{\nu_+}(1-i\xi)^{-\nu_+}\exp\hat{I}^{++}(q,0)+R_+(q,\xi), \tag{D.19}$$

where $R_+(q,\xi)$ admits an estimate: for $k\in\mathbb{Z}_+$,

$$|\partial_q^k R_+(q,\xi)|\le C(1+|\xi|)^{-\nu_++s-\nu}|q|^{-k}. \tag{D.20}$$

For $\phi_q^-(\xi)$, the constructions and results above are modified as follows. Using (2.15)–(2.16), we obtain, for $|q|\ll z^\nu$,

$$\frac{q+\psi(iz-0)}{q+\psi(iz+0)}=\frac{d_-}{d_+}e^{-i\pi\nu}+O(|q|/z^\nu). \tag{D.21}$$

Notice the opposite signs in the exponents in (D.21) and (D.4); this explains the sign in the exponent in the formula below

$$\Phi^-(q,z):=\frac{q+\psi(iz-0)}{q+\psi(iz+0)}e^{i2\pi\nu_-} \tag{D.22}$$

(cf. (D.6)). Next, we introduce

$$\hat{I}^{--}(q,\xi)=\frac{1}{2\pi i}\int_{\lambda_+}^{\infty}\frac{\ln\Phi^-(q,z)dz}{z+i\xi}, \tag{D.23}$$

and then prove that

$$\phi_q^-(\xi) = (1 + i\xi/\lambda_+)^{-\nu_-} \exp[\hat{I}^{--}(q,0) - \hat{I}^{--}(q,\xi)] \tag{D.24}$$

admits the following representation on the set $|q| \le |\xi|^s$:

$$\phi_q^-(\xi) = (\lambda_+)^{\nu_-}(1 + i\xi)^{-\nu_-} \exp \hat{I}^{--}(q,0) + R_-(q,\xi), \tag{D.25}$$

where $R_-(q,\xi)$ admits an estimate: for $k \in \mathbb{Z}_+$,

$$|\partial_q^k R_-(q,\xi)| \le C(1 + |\xi|)^{-\nu_- + s - \nu}|q|^{-k} \tag{D.26}$$

D.5. **Calculation of the leading term of asymptotics.** We repeat the same steps as in the case $\nu > 1$. The proofs of bounds (4.11), (4.12) and (4.13) for functions $\tilde{G}(q,x)$ and $\tilde{G}(q,0+)$, the absolute convergence of the integral in (3.10) in §4.2 for $|k| = 1$, and justification of the reduction procedure in §4.3 are based on (4.7)-(4.8) and (4.9)-(4.10). In the case case under consideration $\nu < 1, \mu = 0$, we have even better estimates, with $\epsilon = 0$ in (4.9)-(4.10). Hence, we can repeat the proofs word by word, with only one exception: at Step IV of the reduction procedure, we use new functions $f^0(p,q,x)$, which is natural in view of new representations (D.24) and (D.25) for $\phi_q^-(\xi)$:

$$f^0(p,q,\xi) = q^{-1-p_1}(\lambda_+)^{\nu_-}\partial_q^{p_2}\left(\exp \hat{I}^{--}(q,0)\right)\partial_q^{p_3}\tilde{G}(q,0+)g_{\nu_-}(x), \tag{D.27}$$

and we use the following analog of (4.21)–(4.22): for a fixed $s \in (0,\nu)$ and $|q| \le |\xi|^s$,

$$\left|\partial_q^k\left(\phi_q^\pm(\xi) - (\mp\lambda_\mp)^{\nu_\pm}(1 \mp i\xi)^{-\nu_\pm}\exp[\hat{I}^{\pm\pm}(q,0)]\right)\right| \le C_{s,k}|q|^{\nu_\pm/\nu - k}|\xi|^{-1+s/\nu}. \tag{D.28}$$

The latter estimate is, in fact, equality (D.25) and estimate (D.26).

Thus, we have proved that the leading term of asymptotics is given by (3.10), where in the decomposition implied by the Leibnitz rule, functions $f(p,q,x)$ are replaced with functions $f^0(p,q,x)$.

D.6. **The leading term.** Finally, we use (4.19), and derive the asymptotic formula (1.4), where

$$\kappa(T) = \frac{\lambda_+^{\nu_-}e^{-rT}}{2\pi i\,\Gamma(1+\nu_-)(-T)}\int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q\left(q^{-1}\exp[\hat{I}^{--}(q,0)]\tilde{G}(q,0+)\right)dq. \tag{D.29}$$

We can deform the contour, integrate by parts back and obtain

$$\kappa(T) = \frac{\lambda_+^{\nu_-}e^{-rT}}{2\pi i\,\Gamma(1+\nu_-)}\int_{\partial\Sigma_{\sigma,\theta}} e^{qT}q^{-1}\exp[\hat{I}^{--}(q,0)]\tilde{G}(q,0+)dq. \tag{D.30}$$

To prove the absolute convergence of the integral in (D.29), we need to establish the following estimate

$$|\partial_q^k \exp[\hat{I}^{--}(q,0)]| \le C_k|q|^{\nu_-/\nu - k}, \quad \forall\ q \in \Sigma_{\sigma,\theta}, \tag{D.31}$$

which is similar to (F.11). The proof of (D.31) is similar to the proof of (D.18). We let $\xi = 0$ in (D.17), and note that on $|z|^\nu \le |q|$,

$$\ln \Phi^-(q,z) = 2\pi i \nu_- + O(|z|^\nu/|q|),$$

hence,

$$\frac{1}{2\pi i}\int_{\lambda_+}^{|q|^{1/\nu}} \frac{\ln \Phi^-(q,z)}{z} dz = \nu_- \ln(|q|^{1/\nu}/\lambda_+) + R_-(q), \tag{D.32}$$

where $R_-(q)$ admits the estimate via $C\int_0^{|q|^{1/\nu}} |q|^{-1} z^{\nu-1} dz \le C_1$. On $|z|^\nu \ge |q|$, $\Phi^-(q,z) = O(|q||z|^{-\nu})$, hence,

$$\left|\int_{|q|^{1/\nu}}^{+\infty} \frac{\ln \Phi^-(q,z)}{z} dz\right| \le C\int_{|q|^{1/\nu}}^{+\infty} |q| z^{-1-\nu} dz \le C_1.$$

This proves (D.31) for $k = 0$.

To estimate the derivatives, we note that for $k \ge 1$,

$$\partial_q^k \ln \Phi^-(q,z) = O(|z|^\nu (|q| + |z|^\nu)^{-k-1}), \tag{D.33}$$

therefore

$$|\partial_q^k \hat{I}^{--}(q,0)| \le C\int_1^{+\infty} \frac{z^{\nu-1}\, dz}{(|q| + |z|^\nu)^{k+1}} \le C_1 |q|^{-k}.$$

Thus, (D.31) holds.

Appendix E. Leading term of asymptotics: remaining cases

E.1. **NIG, Case $\mu = 0$.** The important difference with the case $\nu \in (0,1)$ considered in Appendix D is that, in the case under consideration, even if $\sigma > 0$ is large and q remains on the line $\mathrm{Im}\, q = \sigma$, $\arg(q + \psi(\eta))$ can assume any value in $(-\pi, \pi)$ as η varies in the lower half-plane with the cut $i(-\infty, \lambda_-]$. Hence, if we take $\theta > \pi/2$, and try to construct the analytic continuation into the region $\Sigma_{\sigma,\theta}$ with $\pi/2 < \theta < \pi$, then, for $\arg q \in (\pi/2, \theta)$, we can have $q + \psi(\eta) \in (-\infty, 0)$ for some η. Thus, below, q will remain in the half-plane $\mathrm{Re}\, q \ge \sigma$.

For these q, all the arguments of Appendix D.1 for the case $\nu \in (0,1)$ are still applicable with $\nu = 1, d_\pm = \delta, \nu_\pm = 1/2$, although there are a couple of small differences. The first additional subtle point is equation (D.2) (cases "+" and "–" being similar, we consider sign "+"). For $\nu = 1$ and $d_+ = d_- = \delta$, we have $\frac{d_-}{d_+} e^{i\pi\nu} = -1$. Hence, we need to be more explicit than we have been in (D.4) and (D.6):

$$\frac{q + \psi(iz - 0)}{q + \psi(iz + 0)} = \frac{\delta(\lambda_- - z)^{1/2}(\lambda_+ - z)^{1/2} e^{i\pi/2} + q - \delta(-\lambda_-\lambda_+)^{1/2}}{\delta(\lambda_- - z)^{1/2}(\lambda_+ - z)^{1/2} e^{-i\pi/2} + q - \delta(-\lambda_-\lambda_+)^{1/2}}. \tag{E.1}$$

Take $\sigma > \delta(-\lambda_-\lambda_+)^{1/2}$. Then, for $\mathrm{Re}\, q \ge \sigma$, (E.1) is of the form $(a + i(b + c))/(a - i(b - c))$, where $a > 0$, $b > 0$, and $c \in \mathbb{R}$. We have

$$\frac{a + i(b + c)}{a - i(b - c)} = \frac{(a + i(b + c))\,(a + i(b - c))}{a^2 + (b - c)^2} = \frac{a^2 + c^2 - b^2 + 2iab}{a^2 + (b - c)^2},$$

which is in the upper half-plane, and, if $b \to +\infty$, the argument of the fraction tends to π from below. This also proves the analog of (D.4), which is now

$$\frac{q + \psi(iz - 0)}{q + \psi(iz + 0)} = e^{i\pi} + O(|q|/|z|),$$

for $z << -|q|$, and we proceed as in the case $\nu \in (0, 1)$, using $\nu = 1$ and $\nu_\pm = 1/2$, with the following slight change

$$|I_2^+(q, \xi)| \le C \int_{-|q|}^{\lambda_-} \frac{|\xi||z||q|^{-1}}{|z|(|z| + |\xi|)} dz \le C|q|^{-1}|q| = C.$$

All other steps of the derivation of the leading term of asymptotics remains exactly the same, with $\nu = 1$ and $\nu_\pm = 1/2$. The only exception is at the last step: we can use (D.29) but not (D.30).

E.2. **NIG, Case $\mu < 0$.** Contrary to the cases considered above, the cases "+" and "–" are no longer similar, when $\mu \neq 0$. The factor $\phi_q^+(\xi)$ can be treated exactly as in the case $\mu = 0$, with $\nu_\pm = 1/2 \mp \gamma_+/\pi$, where $\gamma_\pm = \arg d_\pm$ and $d_\pm = \delta \mp i\mu$. Indeed, for $\mathrm{Im}\, \eta < 0$, $\mathrm{Re}(-i\mu\eta) > 0$, and, therefore, the presence of an additional term $-i\mu\eta$ in

$$q + \psi(\eta) = q - i\mu\eta + \delta[(\lambda_+ + i\xi)^{1/2}(-\lambda_- - i\xi)^{1/2} - (\lambda_-\lambda_+)^{1/2}]$$

does not require any changes in the proof and statements above. We obtain the same bounds (D.9), (D.12), (D.13) (for sign "+" only) and asymptotic formula (D.19). Using (D.13), we derive the same bounds (4.11), (4.12) and (4.13) for functions $\tilde{G}(q, x)$ and $\tilde{G}(q, 0+)$. To finish the proof of the absolute convergence of the integral in (3.10) in §4.2 for $|k| = 1$, and justification of Steps I-IV in the reduction procedure in §4.3, we need similar bounds and asymptotic formula for $\phi_q^-(\xi)$. However, since $\mu < 0$, we cannot obtain the information about $\phi_q^-(\xi)$ in a similar way. Indeed, we will have to transform the contour of integration to the upper cut $i[\lambda_+, +\infty)$, but then $\mathrm{Re}(-i\mu\eta)$ may become an arbitrarily large negative number. So, we use the Wiener-Hopf factorization formula instead to calculate $\phi_q^-(\xi)$. Choosing $\sigma = \mathrm{Re}\, q$ sufficiently large so that

$$\mathrm{Re}(q + \psi(\xi)) \ge C > 0, \quad \mathrm{Im}\, \xi \in (\lambda_-, \lambda_+),\ \mathrm{Re}\, q \ge \sigma,$$

we set

$$\phi_q^-(\xi) = \frac{q}{(q + \psi(\xi))\phi_q^+(\xi)}. \tag{E.2}$$

Using (E.2) and the estimates and asymptotic formulas for $\phi^+_q(\xi)$ and its derivatives $\partial^k_q\phi^+_q(\xi)$, we derive similar estimates and asymptotic formulas for $\phi^-_q(\xi)$ and its derivatives $\partial^k_q\phi^-_q(\xi)$. We start with the following estimate

$$c(|q|+|\xi|)\le |q+\psi(\xi)|\le C(|q|+|\xi|), \tag{E.3}$$

where $C, c>0$ are independent of q in the half-plane $\mathrm{Re}\, q\ge\sigma$ and ξ in the strip $\mathrm{Im}\,\xi\in[\omega'_-,\omega'_+]$. Since $\nu_++\nu_-=1$, (D.9) with sign "+" and (E.3) imply (D.9) with sign "–" but for q in the half-plane $\mathrm{Re}\, q\ge\sigma$ and ξ in the strip $\mathrm{Im}\,\xi\in[\omega'_-,\omega'_+]$ only. The estimates and asymptotic formula for $\phi^-_q(\xi)$ below will be valid under the same additional restriction. This will suffice for calculation of the leading term of the asymptotics of the price although a certain additional argument will be needed.

We have

$$\partial^k_q\frac{q}{q+\psi(\xi)}=(-1)^{k-1}k!\frac{\psi(\xi)}{(q+\psi(\xi))^{k+1}}=O\left(\frac{|\xi|}{(|q|+|\xi|)^{k+1}}\right),$$

and

$$\partial_q(1/\phi^+_q(\xi))=-\partial_q I^+(q,\xi)/\phi^+_q(\xi),$$

therefore, using the Leibnitz rule and (D.12) and (D.13) with sign "+", we prove (D.13) with sign "–". Next, we derive the leading term of asymptotics of $\phi^-_q(\xi)$ as $\mathrm{Re}\,\xi\to\pm\infty$ in the strip and $|q|\le|\xi|^s$, $s\in(0,1)$. We have

$$\frac{q}{q+\psi(\xi)}=\frac{q}{d_\pm}|\xi|^{-1}+O(|\xi|^{s-1})=\frac{q}{d}(1+i\xi)^{-\nu_-}(1-i\xi)^{-\nu_+}+O(|\xi|^{s-1}).$$

Therefore, using (D.19), we obtain

$$\phi^-_q(\xi)=\frac{q}{d}(-\lambda_-)^{\nu_+}(1+i\xi)^{-\nu_-}\exp[-\hat{I}^{++}(q,0)]+R_-(q,\xi), \tag{E.4}$$

where $R_-(q,\xi)$ admits an estimate: for $k\in\mathbb{Z}_+$,

$$|\partial^k_q R_-(q,\xi)|\le C(1+|\xi|)^{-\nu_++s-\nu}|q|^{-k}. \tag{E.5}$$

The estimates obtained above allow us to prove the convergence of the integral in (3.10) in §4.2 for $|k|=1$, and justify the reduction procedure exactly as in the cases considered above. There is one subtle difference, however. Since the estimates and asymptotic formula for $\phi^-_q(\xi)$ are valid for ξ in a strip around the real axis rather than in the half-plane $\mathrm{Im}\,\xi\le\omega'_+$, we cannot state that the functions, which appear at different stages of the proof, are supported on $[0,+\infty)$. For instance, instead of (4.24), we can prove only that it only that

$$\int_{\mathrm{Re}\, q=\sigma} e^{qT}(f^{0,s}(p,q,x)-f^{1,s}(p,q,x))dq\in H^{1/2+s_2-\epsilon}(\mathbb{R}), \tag{E.6}$$

for any $\epsilon>0$, where $s_2>\nu_-$. Employing the Sobolev embedding theorem, we can state that integral in (E.6) defines a function of class $C^{s_2-\epsilon}$, for any $\epsilon>0$, but we cannot state that this function vanishes on $(-\infty,0]$ (and hence, it tends to zero as

$x \to 0$ faster than x^{ν_-}). Similarly, in all other cases, we derive estimates in H^s-spaces rather than $\mathring{H}^s$-spaces. In the end, we can state that the option price admit the representation

$$V(T,x) = \kappa(T)\mathbb{1}_{(0,+\infty)}x^{\nu_-}e^{-x} + V_1(T,x), \tag{E.7}$$

where $V_1(T,\cdot) \in H^{\nu_-+s'}(\mathbb{R})$ for some $s' > \nu_-$, and

$$\kappa(T) = \frac{(-\lambda_-)^{\nu_+}e^{-rT}d^{-1}}{2\pi i\,\Gamma(1+\nu_-)(-T)} \int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q\left(\exp[-\hat{I}^{++}(q,0)]\tilde{G}(q,0+)\right)dq. \tag{E.8}$$

By Sobolev's embedding theorem, $V_1(T,\cdot) \in C^{s'-\epsilon}$, for any $\epsilon > 0$. Since $V(T,x)$ and the first term on the RHS of (E.7) vanish on $(-\infty,0]$, $V_1(T,x)$ also does, and we conclude that the asymptotic formula (1.4) holds, with $\kappa(T)$ given by (E.8).

E.3. **NIG, Case $\mu > 0$.** This is the mirror reflection of Case $\mu < 0$, and should we consider the up-and-out barrier options, the symmetry would had been perfect. Since we stick to the same down-and-out case, the argument above simplifies somewhat because we can derive all necessary bounds and asymptotic formula for $\phi_q^-(\xi)$ as we did in Case $\mu = 0$. Hence, the resulting asymptotic formula is exactly as in Case $\mu = 0$ (only with $\nu_- \neq 1/2$).

E.4. **Case $\nu \in [0+,1)$, $\mu < 0$.** Here $\nu = 0+$ stands for VG. The estimates below are derived for $\nu \in (0,1)$. Since only the property $|\psi(\xi) + i\mu\xi| \le C(1+|\xi|^\nu)$ is used, the same estimates work in the VG case, with arbitrarily small $\nu > 0$.

E.4.1. *Estimates and bounds for $\phi_q^+(\xi)$.* As in the case $\nu \in (0,1)$, $\mu = 0$, we can reduce the calculation of $I^+(q,\xi)$ to the integration over the cut $i(-\infty,\lambda_-]$ and obtain the analog of (D.2). Since $\mu < 0$ and $z \le \lambda_- < 0$, we have

$$\Phi(q,z) := \frac{q+\psi(iz-0)}{q+\psi(iz+0)} = 1 + O\left(\frac{|z|^\nu}{|q|+|z|}\right).$$

and $\partial_q^k\Phi(q,z) = O(|z|^\nu(|q|+|z|)^{-k-1}$. Therefore, estimation of the integral (D.2) simplifies. We have

$$\begin{aligned} |\partial_q^k I^+(q,\xi)| &\le C\int_{-\infty}^{-\lambda_-} \frac{|\xi||z|^\nu dz}{|z|(|q|+|z|)^{k+1}(|q|+|\xi|)} \le C_1\int_{-\infty}^{-\lambda_-} \frac{|z|^{\nu-1}dz}{(|q|+|z|)^{k+1}} \\ &\le C_2\left[\int_{-|q|}^{-\lambda_-} |q|^{-1-k}|z|^{\nu-1}dz + \int_{-\infty}^{-|q|} |z|^{\nu-2-k}dz\right]. \end{aligned}$$

Hence,

$$|\partial_q^k I^+(q,\xi)| \le C|q|^{\nu-1-k}, \tag{E.9}$$

$$c \le |\phi_q^+(\xi)| \le C, \tag{E.10}$$

where $C, c > 0$, and, for $k = 1, 2, \dots$,

$$|\partial_q^k \phi_q^+(\xi)| \le C|q|^{\nu-1-k}. \tag{E.11}$$

Next, $\phi_q^+(\xi) = \exp[\hat{I}^+(q,0) - \hat{I}^+(q,\xi)]$, where

$$\hat{I}^+(q,\xi) = \frac{1}{2\pi i}\int_{-\infty}^{\lambda_-} \frac{\ln\Phi(q,z)}{z+i\xi}dz.$$

We have

$$|\partial_q^k \hat{I}^+(q,\xi)| \le C\int_{-\infty}^{\lambda_-} \frac{|z|^\nu dz}{(|z|+|\xi|)(|z|+|q|)^{k+1}}.$$

If $|\xi| \ge |q|$, then

$$\begin{aligned}|\partial_q^k \hat{I}^+(q,\xi)| &\le C\left[\int_{-|q|}^{\lambda_-} + \int_{-|\xi|}^{-|q|} + \int_{-\infty}^{-|\xi|}\right]\frac{|z|^\nu dz}{(|z|+|\xi|)(|z|+|q|)^{k+1}} \\ &\le C\left[|\xi|^{-1}|q|^{-1-k}\int_{-|q|}^{\lambda_-}|z|^\nu dz + |\xi|^{-1}\int_{-|\xi|}^{-|q|}|z|^{\nu-1-k}dz + \int_{-\infty}^{-|\xi|}|z|^{\nu-2-k}dz\right] \\ &\le C_1\left[|\xi|^{-1}|q|^{\nu-k} + |\xi|^{-1+\nu}|q|^{-k} + |\xi|^{-1+\nu-k}\right] \le C_2|\xi|^{-1+\nu}|q|^{-k}.\end{aligned}$$

Similarly, if $|\xi| \le |q|$, then $|\partial_q^k \hat{I}^+(q,\xi)| \le C_2|q|^{-1+\nu-k}$. Thus, in all cases,

$$|\partial_q^k \hat{I}^+(q,\xi)| \le C_k(|\xi|+|q|)^{\nu-1}|q|^{-k}. \tag{E.12}$$

It follows that

$$\phi_q^+(\xi) = \exp\hat{I}^+(q,0) + R_+(q,\xi), \tag{E.13}$$

where $R_+(q,\xi)$ admits an estimate

$$|\partial_q^k R_+(q,\xi)| \le C_k(|\xi|+|q|)^{\nu-1}|q|^{-k} \tag{E.14}$$

(with C_k different from the ones in (E.12)), for $\mathrm{Re}\, q \ge \sigma$ and $\mathrm{Im}\,\xi \ge \omega'_-$.

Note that (E.13) gives the leading term of asymptotics of $\phi_q^+(\xi)$ as $\xi \to \infty$ in the half-plane $\mathrm{Im}\,\xi \ge \omega'_-$, uniform w.r.t. q in the half-plane $\mathrm{Re}\, q \ge \sigma$, whereas in the previous cases, we were able to derive an asymptotic formula in the region $|q| \le |\xi|^s$, $s \in (0,\nu)$ only.

E.4.2. *Estimates for $\tilde{G}(q,x)$ and $\tilde{G}(q,0+)$.* Due to the presence of the constant term in (E.13), we cannot state, as we did in the previous cases, that, if (3.1) holds, then, for each q, $\phi_q^+(\xi)\hat{G}(\xi) \in L_1$ and, hence, $\tilde{G}(q,x)$ is continuous; we also cannot state that $(1+iD)\tilde{G}(q,\cdot) \in H^{\nu_+-1/2-\epsilon}(\mathbb{R})$, for arbitrary $\epsilon > 0$. Moreover, since in the case under consideration, $\nu_+ = 0$, this will not allow us to state that $\mathbb{1}_{(0,+\infty)}(1+iD)\tilde{G}(q,\cdot) \in H^{\nu_+-1/2-\epsilon}(\mathbb{R})$ without additional conditions and argument.

First, we impose a stronger condition on G, which excludes digital down-and-out options, namely,

$$|\hat{G}(\xi)| \le C(1+|\xi|)^{-\rho}, \tag{E.15}$$

where $\rho > 1$. Then we can obtain sufficient bounds for $\tilde{G}(q,x), (1+iD)\tilde{G}(q,\cdot)$ and $\tilde{G}(q,0+)$, and their derivatives w.r.t. q, as in the basic case $\nu > 1$. We start with the representation

$$\tilde{G}(q,x) = \exp[\hat{I}^+(q,0)]G(x) + R_+(q,D)G(x). \tag{E.16}$$

From (E.12), $\exp[\hat{I}^+(q,0)]$ is bounded, and, for $k \ge 1$,

$$|\partial_q^k \exp[\hat{I}^+(q,0)]| \le C_k|q|^{-k-1+\nu}. \tag{E.17}$$

Hence, $\exp[\hat{I}^+(q,0)]G(0+)$ admits the same estimates (which are better than (4.13) with $\epsilon = 0$); for $R_+(q,D)G(0+)$, these estimates follow from (E.13)–(E.14), and we conclude that $\tilde{G}(q,0+)$ satisfies these estimates as well. Next, (3.11) is justified if $(1+iD)\mathcal{E}_q^+G \in H^s(\mathbb{R})$, for some $s \in (-1/2,1/2)$; in our case, we can take any $s \in (-1/2,-1/2+\rho-\epsilon)$. In the reduction scheme, it was also important that, for any $\epsilon > 0$, s can be chosen so that $s > -1/2+\nu_+-\epsilon$. In our case, $\nu_+ = 0$, hence, the last condition is satisfied.

Now consider digitals, for instance, the digital put with $G(x) = \mathbb{1}_{(-\infty,\ln K)}(x)$, $K > 1$. Since $G(0+) = 1$ is well-defined, $\exp[\hat{I}^+(q,0)]G(0+) = \exp[\hat{I}^+(q,0)]$; $R_+(q,D)G(x)$ admit the same estimates as $\exp[\hat{I}^+(q,0)]$. (Generalized) function $(1+iD)\mathcal{E}_q^+G$ can be represented in the form $\exp[\hat{I}^+(q,0)]\delta_{\ln K}+v(q,x)$, where v is as regular as we need it to be, and $\delta_{\ln K}$ is the Dirac delta-function supported at $\ln K$. The $\delta_{\ln K}$ does not influence the price on $(0,\ln K)$ because in the formula for the price, $\delta_{\ln K}$ will appear in the form $\mathcal{E}_q^-\delta_{\ln K}$, which is a function supported on $[\ln K,+\infty)$.

We omit term $\mathcal{E}_q^-\delta_{\ln K}$ as irrelevant for calculation of the asymptotics of the price and get the following estimate

$$|\partial_q^k \widehat{\mathbb{1}_{(0,+\infty)}\tilde{G}}(q,\xi)| \le C_k(1+|\xi|)^{-1}|q|^{-k}. \tag{E.18}$$

E.4.3. *Estimates and bounds for $\phi_q^-(\xi)$.* As in the case $\nu = 1$, $\mu < 0$, we use the Wiener-Hopf factorization formula

$$\phi_q^-(\xi) = \frac{1}{\phi_q^+(\xi)}\frac{q}{q+\psi(\xi)}. \tag{E.19}$$

From (E.10)–(E.14), the first factor $1/\phi_q^+(\xi)$ is (uniformly) bounded, and its derivatives admit estimates

$$|\partial_q^k(1/\phi_q^+(\xi))| \le C_k|\xi|^{\nu-1}|q|^{-k}, \qquad k = 1,2,\dots; \tag{E.20}$$

furthermore, from (E.10) and (E.13)

$$1/\phi_q^+(\xi) = \exp[-\hat{I}^+(q,0)] + R_{+,-}(q,\xi), \tag{E.21}$$

where $R_{+,-}(q,\xi)$ admits an estimate (E.14).

The corresponding estimates for the second factor are worse. If $\nu\in(0,1)$, then, for $k\in\mathbb{Z}_+$,

$$|\partial_q^k(q+\psi(\xi))^{-1}|\le C_k(|q-i\mu\xi|+|\xi|^\nu)^{-1-k}. \tag{E.22}$$

In the case of VG with nonzero drift, $|\xi|^\nu$ must be replaced with $\ln|\xi|$.

For $s\in(0,1)$, the asymptotics of $\phi_q^-(\xi)$ in the region $|q|\le|\xi|^s$ is

$$\phi_q^-(\xi)=q(-\mu)^{-1}\exp[-\hat I^+(q,0)](1+i\xi)^{-1}+O(|\xi|^{\nu-2}). \tag{E.23}$$

E.4.4. *Justification of the Laplace inversion formula.* We integrate by parts in

$$V(T,x)=\frac{1}{2\pi i}\int_{\mathrm{Re}\,q=\sigma}e^{Tq}\,\frac{q}{q+\psi(D)}q^{-1}\phi_q^+(D)^{-1}\mathbb{1}_{(0,\infty)}\phi_q^-(D)G(x)dq \tag{E.24}$$

once or more times. The Leibnitz rule yields a linear combination of integrals with integrands

$$f(p,q,x):=\partial_q^{p_1}((q+\psi(D))^{-1})(\partial_q^{p_2}\phi_q^+(D)^{-1})\mathbb{1}_{(0,\infty)}\partial_q^{p_3}\phi_q^+(D)G.$$

(since we want to take into account the multiplicative structure of (E.19), we modify the definition of functions $f(p,q,x)$ used in the previous cases). In view of (E.22), the Fourier transform of $f(p,q,\xi)$ w.r.t. x admits the following estimate

$$|\hat f(p,q,\xi)|\le C_k(|q-i\mu\xi|+|\xi|^\nu)^{-1-p_1}|q|^{-p_2-p_3}|\xi|^{-1};$$

in the case of VG, $|\xi|^\nu$ needs to be replaced with $\ln|\xi|$. If $p_2+p_3>0$, then $-p_2-p_3$ can be replaced with a better estimate $-p_2-p_3-1+\nu$. It follows that, if $|p|\ge1$, then the integral in the inverse Laplace transform formula with $f(p,q,x)$ defines a continuous function of T, with values in class $H^s(\mathbb{R})$, for any $s<1/2+\nu$. If $\nu>0$, this gives continuity of the price w.r.t. x; in the case of VG, one should use (A.2), as in the reduction scheme for calculation of the leading term of asymptotics of the price. As we will see below, in the case of VG, the continuity is the best result we are able to obtain.

E.4.5. *Calculation of the leading term.* The reduction scheme used in the previous cases need to be modified, because the proof of (4.23) with $s_2>\nu_-=1$ becomes possible only if $|p|$ satisfies $p_1\nu\ge1$, and, in addition, the analytical expression for $f^0(p,q,x)$ is somewhat different. As in all the cases above, this expression is dictated by the asymptotic formula for $\phi_q^-(\xi)$, which is now (E.23). So, we define

$$f^0(p,q,\xi)=(-\mu)^{-1}\partial_q^{p_2}(\exp[-\hat I^+(q,0])\partial_q^{p_3}\tilde G(q,0+)(1+i\xi)^{-2},$$

and follow the next two steps. First, given $s\in(0,1)$, we prove that if p_2+p_3 is sufficiently large, then $\mathbb{1}_{|\xi|^s\le|q|}f(p,q,\xi)$ admits estimate (4.23). If $\nu>0$, then, with each differentiation, the rate of decay w.r.t. q increases by ν, at least (see (E.22), (E.20), (E.18)), hence, given N, we can find $k>0$ such that if $|p|\ge k$, then

$$|\widehat{f}(p,q,\xi)|\le C_p(1+|\xi|)^{-N}|q|^{-N},$$

which proves (4.23). In the case of VG, the best we can get is

$$|\widehat{f}(p,q,\xi)| \le C_p(1+|\xi|)^{-1}(\ln(1+|\xi|))^{-N}|q|^{-N},$$

which can be used only for a proof of the continuity of remainder terms in the asymptotic formula for the price but not for the proof that the rate of decay of the remainder is greater than $1(=\nu_-)$. So, the next argument, although applicable to the case of VG as well, cannot improve the weak result in the VG case.

Next, given $s \in (0,1)$, we prove that if $|p|$ is sufficiently large, then

$$|(\widehat{f}(p,q,\xi) - \widehat{f}^2(p,q,\xi))\mathbb{1}_{|\xi|^s \ge |q|}| \le C|q|^{-s_1'}(1+|\xi|)^{-1-s_2'}, \tag{E.25}$$

where $s' > 1$ and $s_2' > \nu_- = 1$. This is evident because on the set $|q| \le |\xi|^s$, $\psi(\xi) = -\mu(1+i\xi) + O(|\xi|^{\nu-1})$, and $\phi^+(\xi) = 1 + O(|\xi|^{\nu-1})$, hence, (E.25) is immediate from (3.11).

The last step of calculation of the leading term of asymptotics is the same as in all the cases above. In the result, we obtain (1.4) with $\nu_- = 1$ (hence, $\Gamma(1+\nu_-) = 1$) and

$$\kappa(T) = \frac{e^{-rT}}{2\pi i\,(-\mu)(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{qT}\partial_q^k\left(\exp[-\hat{I}^+(q,0)]\tilde{G}(q,0+)\right)dq, \tag{E.26}$$

where $k > 0$ is sufficiently large. The integrand decays as $|q|^{-k+\nu-1}$, hence, the integral absolutely converges if $k \ge 1$. Thus, even though for the proof, a choice of $k > 1$ is necessary, we can integrate by parts back and use (E.26) with $k = 1$.

E.5. **Case $\nu \in (0,1), \mu > 0$.** This time the function $\phi_q^-(\xi)$ is well-behaved. The result is not so pleasant from the point of view of the regularity of the solution $V(T,x)$, because

$$\phi_q^-(\xi) = \exp \hat{I}^-(q,0) + R_-(q,\xi),$$

where

$$\hat{I}^-(q,\xi) = \frac{1}{2\pi i}\int_{\lambda_+}^{+\infty}\frac{\ln\Phi(q,z)}{z+i\xi}dz,$$

and $R_-(q,\xi)$ admits the following bound

$$|\partial_q^k R_-(q,\xi)| \le C_k(|\xi|+|q|)^{\nu-1}|q|^{-k}, \tag{E.27}$$

for $\operatorname{Re} q \ge \sigma$ and $\operatorname{Im}\xi \le \omega_+'$. Indeed, this leads to the the price being discontinuous at the barrier. However, the proof simplifies. In particular, for $\operatorname{Im}\xi \in [\omega_-', \omega_+']$,

$$\phi_q^+(\xi) = \frac{q}{q+\psi(\xi)}\phi_q^-(\xi)^{-1} = O\left(\frac{|q|}{(|q|+|\xi|)^\nu}\right),$$

therefore, if $\nu > 0$ or $|\hat{G}(\xi)| \le C(1+|\xi|)^{-\rho}$, where $\rho > 0$, then $\tilde{G}(q,x)$ is continuous w.r.t. x, and we obtain the result

$$V(T,0+) = \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\operatorname{Re} q=\sigma} e^{Tq}\,\partial_q^k\left(q^{-1}\exp[\hat{I}^-(q,0)]\tilde{G}(q,0+)\right)dq.$$

It can be shown that it suffices to differentiate only once to obtain an absolutely converging integral.

Appendix F. Technical proofs

F.1. **Proof of Lemma 4.1.** (i) It follows from the asymptotic properties (2.15)–(2.18) of $\psi(\rho e^{i\varphi})$, that there exist $\tilde\theta > 0$ and $\tilde\gamma \in (0, \frac{\pi}{2})$ such that if $\varphi \in [-\tilde\theta, \tilde\theta]$ and $\rho > 0$ is sufficiently large, then $\psi(\rho e^{i\varphi}) \in \Sigma_{0,\tilde\gamma}$. Next, for any $\theta \in (\pi/2, \pi - \tilde\gamma)$, there exist $\rho_0 \geq 0$ and $\delta > 0$ such that

$$q + \psi(\rho e^{i\varphi}) \in \Sigma_{0,\pi-\delta} \setminus \{0\}, \tag{F.1}$$

for any $q \in \Sigma_{0,\theta}$, $\varphi \in [-\tilde\theta, \tilde\theta]$, $\rho \geq \rho_0$. If we replace q with $q + \sigma$, where $\sigma > 0$, then the LHS in (F.1) will shift to the right, and, therefore, (F.1) will remain valid for the same ρ_0, θ, $\tilde\theta$. Finally, if $0 \leq \rho \leq \rho_0$, then, for sufficiently large σ, (F.1) holds for any $q \in \Sigma_{\sigma,\theta}$ and any φ.

(ii), (iii) It suffices to take $\tilde\theta \in (\pi/2, \min(\pi - \tilde\gamma, \pi\nu/2))$.

(iv) Estimates for signs "+" and "–" being similar, we consider sign "+". First, we transform the line of integration in (4.2)

$$I^+(q,\xi) = \frac{1}{2\pi i}\int_{\mathcal{L}_{\omega_-,\tilde\theta}} \frac{\xi \ln \Psi(q,\eta)}{\eta(\xi-\eta)} d\eta, \tag{F.2}$$

where $\mathcal{L}_{\omega_-,\tilde\theta} := \{i\omega_- + \rho e^{i\varphi} | \rho \geq 0, \varphi = -\pi + \tilde\theta \ \text{or} \ \varphi = -\tilde\theta\}$. The advantage of this transformation is that there exists a constant $C > 0$ such that

$$|\xi - \eta|^{-1} \leq C(|\xi| + |\eta|)^{-1}, \quad \operatorname{Im}\xi \geq \omega'_-, \eta \in \mathcal{L}_{\omega_-,\tilde\theta}. \tag{F.3}$$

Next, we prove that there exists a constant $C > 0$ such that for $q \in \Sigma_{\sigma,\theta}$ and $\eta \in \mathcal{L}_{\omega_-,\tilde\theta}$

$$|\ln \Psi(q,\eta)| \leq C \begin{cases} |\eta|^\nu/|q|, & |\eta|^\nu \leq |q| \\ |q|/|\eta|^\nu, & |\eta|^\nu \geq |q|. \end{cases} \tag{F.4}$$

From (4.1), $\sup_{|\eta|^\nu \leq c|q|} |\Psi(q,\eta) - 1| \to 0$ as $c \to 0$, and, similarly to (2.19),

$$\sup_{|\eta|^\nu \geq C|q|} |\Psi(q,\eta) - 1| \to 0 \ \text{as} \ C \to +\infty.$$

Since $\ln(1 + x) = O(x)$ as $x \to 0$, we conclude that (F.4) holds if $|\eta|^\nu \leq c|q|$ or $|\eta|^\nu \geq C|q|$, where $c > 0$ is sufficiently small and $C > 0$ is sufficiently large. Since $\Psi(q,\eta)$ is uniformly bounded, (F.4) holds for $c \leq |\eta|^\nu/|q| \leq C$ as well.

Now, we divide $\mathcal{L}_{\omega_-,\tilde\theta_-}$ in (F.2) into two pieces by conditions $|\eta|^\nu \leq |q|$ and $|\eta|^\nu \geq |q|$, and denote by $I^+_j(q,\xi), j = 1, 2$, the corresponding integrals. Using (F.3) and

(F.4), we obtain

$$\text{(F.5)}\quad |I_1^+(q,\xi)| \le C_2\int_1^{|q|^{1/\nu}} \frac{|\xi||\eta|^\nu/|q|}{|\eta|(|\eta|+|\xi|)}d\eta \le C_2|q|^{-1}\int_1^{|q|^{1/\nu}} |\eta|^{\nu-1}d\eta \le \frac{C_2}{\nu},$$

$$\text{(F.6)}\quad |I_2^+(q,\xi)| \le C_3\int_{|q|^{1/\nu}}^{\infty} \frac{|\xi||q||\eta|^{-\nu}}{|\eta|(|\eta|+|\xi|)}d\eta \le C_3|q|\int_{|q|^{1/\nu}}^{\infty} |\eta|^{-\nu-1}d\eta \le \frac{C_2}{\nu}.$$

Estimates (F.5) and (F.6) prove (4.7), and (4.8) is immediate from (4.7) and (iii).

(v) Bound (4.10) is immediate from (4.9), therefore, it suffices to prove the latter. Using the multiplicative structure of $\Psi(q,\eta)$ and the Leibnitz rule, it is straightforward to show that, for $k \ge 1$,

$$\text{(F.7)}\qquad \left|\partial_q^k(\ln\Psi(q,\eta))\right| \le C_k|q|^{-k}(1+|\eta|/|q|^{1/\nu})^{-1},$$

therefore, for any $\epsilon > 0$,

$$|\partial_q^k I^\pm(q,\xi)| \le C_k|q|^{-k+\epsilon}\int_1^\infty \frac{|\xi|d\eta}{|\eta|^{1+\nu\epsilon}(|\xi|+|\eta|)} \le C_{k,\epsilon}|q|^{\epsilon-k}.$$

This finishes the proof of the lemma.

F.2. **Proof of Lemma 4.2.** Cases "+" and "–" being similar, we consider sign "+". Deforming the line of integration in (4.3) to the same contour $\mathcal{L}_{\omega_-,\tilde\theta}$ as in Lemma 4.1, we obtain

$$\text{(F.8)}\qquad \hat I^+(q,\xi) = \pm\frac{1}{2\pi i}\int_{\mathcal{L}_{\omega_-,\tilde\theta}} \frac{\ln\Psi(q,\eta)}{\eta-\xi}d\eta,$$

We divide the contour of integration in (F.8) into three parts: $|\eta| \le |q|^{1/\nu}$, $|q|^{1/\nu} \le |\eta| \le |\xi|$, and $|\eta| \ge |\xi|$. Denote by $\hat I_j^\pm(q,\xi)$ the corresponding integrals. On the strength of (F.4),

$$|\hat I_1^+(q,\xi)| \le C\int_1^{|q|^{1/\nu}} \frac{|\eta|^\nu|q|^{-1}}{|\xi|}d\eta \le C_1|\xi|^{-1}|q|^{-1+\frac{1+\nu}{\nu}} \le C_1|\xi|^{-1+s/\nu},$$

and, since $\nu > 1$,

$$|\hat I_2^+(q,\xi)| \le C\int_{|q|^{1/\nu}}^{|\xi|} \frac{|q||\eta|^{-\nu}}{|\xi|}d\eta \le C_1\frac{|q|}{|\xi|}|\xi|^{1-\nu} \le C_2|q||\xi|^{-\nu} \le C_3|\xi|^{-1+s/\nu},$$

$$|\hat I_3^+(q,\xi)| \le C\int_{|\xi|}^{\infty} \frac{|q||\eta|^{-\nu}}{|\eta|}d\eta \le C_1|q||\xi|^{-\nu} \le C_3|\xi|^{-1+s/\nu}.$$

This proves (4.20) for $k = 0$. If $k > 0$, we differentiate under the integral sign in (F.8), use (F.7), and divide the contour of integration into the same three parts. Denote by $I_{k,j}^+(q,\xi)$, $j = 1,2,3$, the corresponding integrals. On the strength of (F.4),

$$|\hat I_{k,1}^+(q,\xi)| \le C\int_1^{|q|^{1/\nu}} \frac{d\,\eta}{(|q|+|\eta|^\nu)^k(|\xi|+|\eta|)}d\eta \le C_1|\xi|^{-1}|q|^{-k+1/\nu} \le C_1|\xi|^{-1+s/\nu}|q|^{-k},$$

$$|\hat{I}^{+}_{k,2}(q,\xi)| \le C\int_{|q|^{1/\nu}}^{|\xi|}\frac{|\eta|^{-k\nu}}{|\xi|}d\eta \le C_1|\xi|^{-1}|q|^{1/\nu-k} \le C_1|\xi|^{-1+s/\nu}|q|^{-k},$$

$$|\hat{I}^{+}_{k,3}(q,\xi)| \le C\int_{|\xi|}^{\infty}|\eta|^{-1-k\nu}d\eta \le C_1|\xi|^{-k\nu} \le C_3|\xi|^{-1+s/\nu}|q|^{-k},$$

because $\nu > 1$.

To prove (4.21), recall that

$$\phi^{\pm}_q(\xi) = \left(\frac{(q/d)^{1/\nu}}{(q/d)^{1/\nu}\mp i\xi}\right)^{\nu_\pm}\exp[\hat{I}^{\pm}(q,0)-\hat{I}^{\pm}(q,\xi)].$$

In view of this equality, it remains to prove that $\exp[\hat{I}^{\pm}(q,\xi)]-1$ and its derivatives admit the same bounds as $\hat{I}^{\pm}(q,\xi)$ and its derivatives (with, possibly, different constants), which is evident, and that

$$\left|\partial_q^k\left(\left(\frac{(q/d)^{1/\nu}}{(q/d)^{1/\nu}\mp i\xi}\right)^{\nu_\pm}-\left(\frac{(q/d)^{1/\nu}}{1\mp i\xi}\right)^{\nu_\pm}\right)\right| \le C_{s,k}|q|^{\nu_\pm/\nu-k}|\xi|^{-1+s/\nu}, \tag{F.9}$$

which is easy to verify using

$$(q/d)^{1/\nu}\mp i\xi = (1\mp i\xi)(1+O(|q|^{1/\nu}/|\xi|)) = (1\mp i\xi)(1+O(|\xi|^{-1+s/\nu}))$$

F.3. **Step V.** To show that we can replace $f^{0,s}(p,q,x)$ with $f^0(p,q,x)$, we need estimates

$$|\partial_q^k(q^{-1}(q/d)^{\nu_-/\nu}\exp[\hat{I}^{+}(q,0)]\tilde{G}(q,0+))| \le C_{k,\epsilon}|q|^{-k-1+\nu_-/\nu+\epsilon}. \tag{F.10}$$

Since we have (4.13), it remains to derive estimates for $\exp[\hat{I}^{+}(q,0)]$. A straightforward modification of estimates (F.5) and (F.6) gives the uniform boundedness of $\hat{I}^{\pm}(q,\xi)$, and an estimate for the derivatives of $\hat{I}^{\pm}(q,\xi)$

$$|\partial_q^k\hat{I}^{\pm}(q,\xi)| \le C_{k,\epsilon}|q|^{-k+\epsilon},\quad k=1,2,\ldots, \tag{F.11}$$

for any $\epsilon > 0$, can be derived similarly to the estimate for the derivatives of $I^{\pm}(q,\xi)$ in part (v) of Lemma 4.1. We use (F.7) and modify estimates (F.5) and (F.6):

$$\begin{aligned}\int_1^{|q|^{1/\nu}}\frac{d\eta}{(|q|+|\eta|^\nu)^k(|\xi|+|\eta|)} &\le C|q|^{-k}\int_1^{|q|^{1/\nu}}\eta^{-1}d\,\eta \le C|q|^{-k}\ln|q|,\\ \int_{|q|^{1/\nu}}^{+\infty}\frac{d\eta}{(|q|+|\eta|^\nu)^k(|\xi|+|\eta|)} &\le C\int_{|q|^{1/\nu}}^{+\infty}\eta^{-k\nu-1}d\,\eta \le C_1|q|^{-k}.\end{aligned}$$

Finally, notice that since $\nu_-/\nu < 1$, (F.10) implies the absolute convergence of integral (4.25).

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